Screening and Adverse Selection in Frictional Markets

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Abstract

We incorporate a search-theoretic model of imperfect competition into an otherwise standard model of asymmetric information with unrestricted contracts. We develop a methodology that allows for a sharp analytical characterization of the unique equilibrium, and then use this characterization to explore the interaction between adverse selection, screening, and imperfect competition. We show how the structure of equilibrium contracts—and hence the relationship between an agent’s type, the quantity he trades, and the corresponding price—is jointly determined by the severity of adverse selection and the concentration of market power. This suggests that quantifying the effects of adverse selection requires controlling for the market structure. We also show that increasing competition and reducing informational asymmetries can be detrimental to welfare. This suggests that recent attempts to increase competition and reduce opacity in markets that suffer from adverse selection could potentially have negative, unforeseen consequences.

Keywords: Adverse Selection, Imperfect Competition, Screening, Transparency, Search Theory

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1 Introduction

Many large and important markets suffer from adverse selection, including the markets for insurance, credit, and certain financial securities. There is mounting evidence that many of these markets also feature some degree of imperfect competition.¹ And yet, perhaps surprisingly, the effect of imperfect competition on prices, allocations, and welfare in markets with adverse selection remains an open question.

Answering this question is important for several reasons. For one, many empirical studies attempt to quantify the effects of adverse selection in the markets mentioned above.² A natural question is to what extent these estimates—and the conclusions that follow—are sensitive to the assumptions being imposed on the market structure. There has also been a recent push by policymakers to make several of the markets mentioned above more competitive and less opaque.³ Again, a crucial, but seemingly underexplored question is whether these attempts to promote competition and reduce information asymmetries are necessarily welfare-improving.

Unfortunately, the ability to answer these questions has been constrained by a shortage of appropriate theoretical frameworks.⁴ A key challenge is to incorporate nonlinear pricing schedules—which are routinely used to screen different types of agents—into a model with asymmetric information and imperfect competition. This paper delivers such a model: we develop a novel, tractable framework of adverse selection, screening, and imperfect competition.

The key innovation is to introduce a search-theoretic model of imperfect competition (à la Burdett and Judd, 1983) into an otherwise standard model with asymmetric information and nonlinear contracts. Within this environment, we provide a full analytical characterization of the unique equilibrium, and then use this characterization to study both the positive and normative issues highlighted above.

First, we show how the structure of equilibrium contracts—and hence the relationship between an agent’s type, the quantity that he trades, and the corresponding price—is jointly determined by the severity of the adverse selection problem and the degree of imperfect competition. In particular, we

¹For evidence of market power in insurance markets, see Brown and Goolsbee (2002), Dafny (2010), and Cabral et al. (2014); Einav and Levin (2015) provide additional references, along with a general discussion. For evidence of market power in various credit markets, see, e.g., Ausubel (1991), Calem and Mester (1995), Petersen and Rajan (1994), Scharfstein and Sunderam (2013), and Crawford et al. (2015). In over-the-counter financial markets, a variety of data suggests that dealers extract significant rents; indeed, this finding is hard-wired into the workhorse models of this market, such as Duffie et al. (2005) and Lagos and Rocheteau (2009).

²See the seminal paper by Chiappori and Salanie (2000), and Einav et al. (2010a) for a comprehensive survey.

³Increasing competition and transparency in health insurance markets is a cornerstone of the Affordable Care Act, while the Dodd-Frank legislation addresses similar issues in over-the-counter financial markets. In credit markets, on the other hand, legislation has recently focused on restricting how much information lenders can demand or use from borrowers.

⁴As Chiappori et al. (2006) put it, “there is a crying need for [a model] devoted to the interaction between imperfect competition and adverse selection.”
show that equilibrium offers separate different types of agents when competition is relatively intense or adverse selection is relatively severe, while they typically pool different types of agents in markets where principals have sufficient market power and adverse selection is sufficiently mild. Second, we explore how total trading volume—which, in our environment, corresponds to the utilitarian welfare measure—responds to changes in the degree of competition and the severity of adverse selection. We show that increasing competition or reducing informational asymmetries is only welfare-improving in markets in which both market power is sufficiently concentrated and adverse selection is sufficiently severe.

Before explaining these results in greater detail, it is helpful to lay out the basic building blocks of the model. The agents in our model, whom we call “sellers,” are endowed with a perfectly divisible good of either low or high quality, which is the seller’s private information. The principals, whom we call “buyers,” offer menus containing price-quantity combinations to potentially screen high and low-quality sellers. Sellers can accept at most one contract, i.e., contracts are exclusive. To this otherwise canonical model of trade under asymmetric information, we introduce imperfect competition by endowing the buyers with some degree of market power. The key assumption is that each seller receives a stochastic number of offers, with a positive probability of receiving only one. This implies that, when a buyer formulates an offer, he understands that it will be compared to an alternative offer with some probability—which we denote by $\pi$—and it will be the seller’s only option with probability $1 - \pi$. This formulation allows us to capture the perfectly competitive case (a la Rothschild and Stiglitz (1976)) by setting $\pi = 1$, the monopsony case (a la Stiglitz (1977)) by setting $\pi = 0$, and everything in between.

For the general case of imperfect competition, with $\pi \in (0, 1)$, the equilibrium involves buyers mixing over menus according to a nondegenerate distribution function. Since each menu is comprised of two price-quantity pairs (one for each type), this implies that the main equilibrium object is a probability distribution over four-dimensional offers. A key contribution of our paper is developing a methodology that allows for a complete, yet tractable, characterization of this complicated equilibrium object.

We begin by showing that any menu can be summarized by the indirect utilities it offers to sellers of each type, which reduces the dimensionality of the distribution from four to two. Next, we establish an important property: in any equilibrium, all menus that are offered by buyers are ranked in exactly the

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5The use of the labels “buyers” and “sellers” is merely for concreteness and corresponds most clearly with an asset market interpretation. These monikers can simply be switched in the context of an insurance market, so that the “buyers” of insurance are the agents with private information and the “sellers” of insurance are the principals.

6Mixing is to be expected for at least two reasons. First, this is a robust feature of nearly all models in which buyers are both monopsonists and Bertrand competitors with some probability, even without adverse selection or non-linear contracts. Second, even in perfectly competitive markets, it is well known that pure strategy equilibria may not exist in an environment with both adverse selection and non-linear contracts.
same way by both low- and high-quality sellers. This property, which we call “strictly rank-preserving,” implies that all equilibrium menus can be ranked along a single dimension. The equilibrium, then, can be described by a distribution function over a unidimensional variable—say, the indirect utility offered to low-quality sellers—along with a strictly monotonic function mapping this variable to the indirect utility offered to the high-quality seller. We show how to solve for these two functions, obtaining a full analytical characterization of all equilibrium objects of interest, and then establish that the equilibrium is unique. Interestingly, our approach not only avoids the well-known problems with existence of equilibria in models of adverse selection and screening, but also requires no assumptions on off-path beliefs to get uniqueness. We then use this characterization to explore the implications of imperfect competition in markets suffering from adverse selection.

First, we show that the structure of menus offered in equilibrium depends on both the degree of competition, captured by $\pi$, and the severity of the adverse selection problem, which is succinctly summarized by a single statistic that is largest (i.e., adverse selection is most severe) when: (i) the fraction of low-quality sellers is large; (ii) the potential surplus from trading with high-quality sellers is small; and (iii) the information cost of separating the two types of sellers, as captured by the difference in their reservation values, is large. Given these summary statistics, we show that separating menus are more prevalent when competition is relatively strong or when adverse selection is relatively severe, while pooling menus are more prevalent when competition is relatively weak and adverse selection is relatively mild. Interestingly, holding constant the severity of adverse selection, the equilibrium may involve all pooling menus, all separating menus, or a mixture of the two, depending on the degree of competition. This finding suggests that attempts to infer the severity of adverse selection from the distribution of contracts that are traded should take into account the extent to which the market is competitive.

Next, we examine our model’s implications for welfare, defined as the objective of a utilitarian social planner. In our context, this objective maps one-for-one to the expected quantity of high-quality goods traded. We first study the relationship between welfare and the degree of competition. Our main finding is that competition can worsen the distortions related to asymmetric information and, therefore, can be detrimental to welfare. When adverse selection is mild, these negative effects are particularly stark: welfare is actually (weakly) maximized under monopsony, or $\pi = 0$. When adverse selection is severe, however, welfare is inverse U-shaped in $\pi$, i.e., an interior level of competition maximizes welfare.

To understand the hump-shape in welfare under severe adverse selection, note that an increase in competition induces buyers to allocate more of the surplus to sellers (of both types) in an attempt to
retain market share. All else equal, increasing the utility offered to low-quality sellers is good for welfare: by relaxing the low-quality seller’s incentive compatibility constraint, the buyer is able to exchange a larger quantity with high-quality sellers. However, *ceteris paribus*, increasing the utility offered to high-quality sellers is bad for welfare: it tightens the incentive constraint and forces buyers to trade less with high-quality sellers. Hence, the net effect of an increase in competition on the quantity of high-quality goods traded depends on whether the share of the surplus offered to high-quality sellers rises faster or slower than that offered to low-quality sellers.

When competition is low, buyers earn a disproportionate fraction of their profits from low-quality sellers. Therefore, when buyers have lots of market power, an increase in competition leads to a faster increase in the utility offered to low-quality sellers, since buyers care relatively more about retaining these sellers. As a result, the quantity traded with high-quality sellers and welfare rise with competition. When competition is sufficiently high, profits come disproportionately from high-quality sellers. In this case, increasing competition induces a faster increase in the utility offered to high-quality sellers and, therefore, a decrease in expected trade and welfare. These results suggest that promoting competition—or policies that have similar effects, such as price supports or minimum quantity restrictions—can have adverse effects on welfare in markets that are sufficiently competitive and face severe adverse selection.

Next, we study the welfare effects of providing buyers with more information—specifically, a noisy signal—about the seller’s type. As in the case of increasing competition, the welfare effects of this perturbation depend on the severity of the two main frictions in the model: imperfect competition and adverse selection. When adverse selection is relatively mild or competition relatively strong, reducing informational asymmetries can actually be detrimental to welfare. The opposite is true when adverse selection and trading frictions are relatively severe. In sum, these normative results highlight how the interaction between these two frictions can have surprising implications for changes in policy (or technological innovations), underscoring the need for a theoretical framework such as ours.

Our baseline model was designed to be as simple as possible in order to focus on the novel interactions between adverse selection and imperfect competition. In Sections 6–8, we analyze several relevant extensions and variants of our model to explore these interactions more deeply and to ensure that our results are robust to alternative specifications, many of which could make our framework more amenable to applied work. In Section 6, we endogenize the level of competition by letting buyers choose the intensity with which they “advertise” their offers, so that the distribution over the number of offers that each seller receives is an outcome rather than a primitive. This allows us to study how the severity of
adverse selection can influence the market structure, and the ensuing welfare implications. In Section 7, we consider a more general market setting with an arbitrary meeting technology, where sellers can meet any number of buyers (including zero). We show how to derive the equilibrium in this setting, using the techniques from our benchmark model, and confirm that our main welfare results hold for certain popular meeting technologies. Finally, in Section 8, we explore a number of additional extensions: we relax the assumption of linear utility to analyze the canonical model of insurance under private information; we allow the degree of competition to differ across sellers of different quality; and we show how to incorporate additional dimensions of heterogeneity, including horizontal and vertical differentiation.

**Literature Review.** Our paper contributes to an extensive body of literature on adverse selection. Our focus on contracts as screening devices puts us in the tradition of Rothschild and Stiglitz (1976), in contrast to the branch of the literature that restricts attention to single price contracts, as in the original model of Akerlof (1970). Most of the literature that studies adverse selection and screening has either assumed a monopolistic or perfectly competitive market structure.\(^\text{7}\)

The main novelty of our analysis is to synthesize a standard model of adverse selection and unrestricted contracts with the search-theoretic model of imperfect competition developed by Butters (1977), Varian (1980), and, in particular, Burdett and Judd (1983). While this model of imperfect competition has been used extensively in both theoretical and empirical work,\(^\text{8}\) to the best of our knowledge none of these papers address adverse selection and screening. A recent paper by Garrett et al. (2014) exploits the Burdett and Judd (1983) model in an environment with screening contracts and asymmetric information, but the asymmetric information is over the agents’ private values. This key difference implies that the role of screening—and how it interacts with imperfect competition—is ultimately very different in our paper and theirs.\(^\text{10}\)

More closely related to our work is the literature that studies adverse selection and nonlinear contracts in an environment with competitive search—most notably the influential paper by Guerrieri et al. (2010).\(^\text{11}\) In that paper, principals post contracts and match bilaterally with agents who direct their search

\(^\text{7}\)For recent contributions to this literature that assume perfectly competitive markets, see, e.g., Bisin and Gottardi (2006), Chari et al. (2014), and Azevedo and Gottlieb (2015).

\(^\text{8}\)For recent examples, see, e.g., Sorensen (2000) and Kaplan and Menzio (2015).

\(^\text{9}\)Carrillo-Tudela and Kaas (2015) analyze a related labor market setting with adverse selection using the on-the-job search model of Burdett and Mortensen (1998), but their focus is quite different from ours.

\(^\text{10}\)In particular, with private values, screening is useful only for rent extraction. Competition reduces (and ultimately, eliminates) these rents and, along with them, incentives to screen. In contrast, with common values, screening plays a central role in mitigating the adverse selection problem. As a result, it disappears only when that problem is sufficiently mild; increased competition serves to strengthen incentives to separate. This interaction is also the source of non-monotonic effects on welfare from increased competition. With private values, on the other hand, welfare unambiguously increases with competition.

\(^\text{11}\)Other papers studying adverse selection with competitive search include Michelacci and Suarez (2006), Kim (2012), Chang
toward specific contracts. A matching technology determines the probability that each agent trades (or is rationed) in equilibrium, as a function of the relative measures of principals offering a specific contract and agents searching for it. As in our paper, Guerrieri et al. (2010) present an explicit model of trade without placing any restrictions on contracts, beyond those arising from the primitive frictions. There are, however, several important differences. The first relates to the role of search frictions. We focus on how perturbations to the search technology affect market power, and study the interaction between the resulting distortions and the underlying adverse selection problem, while Guerrieri et al. (2010) and others focus on the role of search frictions in providing incentives (through the probability of trade) and not on market power per se. Second, depending on parameters, our equilibrium menus can be pooling, separating or a combination of both; the approach in Guerrieri et al. (2010), on the other hand, always leads to separating equilibria. In this sense, our approach has the potential to speak to a richer set of observed outcomes. Finally, we obtain a unique equilibrium without additional assumptions or refinements, whereas uniqueness in Guerrieri et al. (2010) relies on a restriction on off-equilibrium beliefs.\footnote{Another related literature studies adverse selection in dynamic models with search frictions, where separation occurs because agents of different types trade at different points in time. See, e.g., Inderst (2005), Moreno and Wooders (2010), Camargo and Lester (2014), and the references therein. In all of these papers, agents are assumed to trade linear contracts.}

An alternative approach to modeling imperfect competition is through product differentiation, as in Villas-Boas and Schmidt-Mohr (1999) and, more recently, Benabou and Tirole (2015), Veiga and Weyl (2012), Mahoney and Weyl (2014), and Townsend and Zhorin (2014).\footnote{Also see Fang and Wu (2016), who propose a slightly different model of imperfect competition.} Identical contracts offered by principals are valued differently by agents because of an orthogonal attribute, which is interpreted as “distance” in a Hotelling interpretation or “taste” in a random utility, discrete choice framework. This additional dimension of heterogeneity is the source of market power, and changes in competition are induced by varying the importance of this alternative attribute, i.e., by altering preferences. We take a different approach to modeling (and varying) competition, which holds constant preferences and, therefore, the potential social surplus. It is also worth pointing out a few key differences in substantive results, particularly about the desirability of competition. In Benabou and Tirole (2015), a tradeoff from increased competition arises not because of adverse selection per se, but from the need to provide incentives to allocate effort between multiple, imperfectly observable or contractible tasks. In fact, without multi-tasking, competition improves welfare even with asymmetric information. This is also the case in Mahoney and Weyl (2014), where attention is restricted to single-price contracts. Veiga and Weyl (2012) also restrict attention to a single contract, but with endogenous “quality,” and find that welfare is
maximized under monopoly. In our setting, depending on parameters, competition can be beneficial or harmful. Though there are a number of differences between their setup and ours (e.g., multidimensional heterogeneity, the contract space, the equilibrium concept), which precludes a direct comparison, we interpret their results as providing a distinct but complementary insight about the interaction between competition and adverse selection.

The rest of the paper is organized as follows. Section 2 describes our model. Section 3 proves key properties of the equilibrium, followed by its construction in section 4. Section 5 contains implications for welfare and policy. Sections 6–8 explore the extensions discussed above. Section 9 concludes, and all proofs can be found in the Appendix.

2 Model

Agents and Preferences. We consider an economy with two buyers and a unit measure of sellers. Each seller is endowed with a single unit of a perfectly divisible good. Buyers have no capacity constraints, i.e., they can trade with many sellers. A fraction \( \mu_l \in (0, 1) \) of sellers possess a low (l) quality good, while the remaining fraction \( \mu_h = 1 - \mu_l \) possess a high (h) quality good.\(^{14}\) Buyers and sellers derive utility \( v_i \) and \( c_i \), respectively, from consuming each unit of a quality \( i \in \{l, h\} \) good, with \( v_l < v_h \) and \( c_l < c_h \). We assume that there are gains from trading both high- and low-quality goods, i.e., that

\[ v_i > c_i \text{ for } i \in \{l, h\}. \]

Frictions. There are two types of frictions in the market. First, there is asymmetric information: sellers observe the quality of the good they possess while buyers do not, though the probability \( \mu_i \) that a randomly selected good is quality \( i \in \{l, h\} \) is common knowledge. In order to generate the standard “lemons problem,” we focus on the case in which

\[ v_l < c_h. \]

The second type of friction is a search friction: as we describe in detail below, the buyers in our model will make offers, but the sellers will not necessarily sample (or have access to) all offers. In particular, we assume that a fraction \( 1 - p \) of sellers will be matched with—and hence receive an offer from—a single buyer, which we assume is equally likely to be either buyer. The remaining fraction of sellers, \( p \), will be matched with both buyers. A seller can only trade with a buyer if they are matched. Throughout the

\(^{14}\)We consider the case where there are \( N > 2 \) types of sellers in Appendix D.
paper, we refer to sellers who are matched with one buyer as “captive,” since they only have one option for trade, and we refer to those who are matched with two buyers as “noncaptive.”

Given these search frictions, a buyer understands that, conditional on being matched with a particular seller, this seller will be captive with probability \(1 - \pi\) and noncaptive with probability \(\pi\), where

\[
\pi = \frac{p}{\frac{1}{2}(1-p) + p} = \frac{2p}{1+p}.
\]

This formalization of search frictions is helpful for deriving and explaining our key results in the simplest possible manner. For one, it allows us to vary the degree of competition with a single parameter, \(\pi\), nesting monoposony and perfect competition as special cases.\(^\text{15}\) Second, since the current formulation ensures that all sellers are matched with at least one buyer, a change in \(\pi\) varies the degree of competition without changing the potential gains from trade or “coverage” in the market; this is particularly helpful in isolating the effects of competition on welfare. However, it is important to stress that our equilibrium characterization and the ensuing results extend to markets with an arbitrary number of buyers and more general meeting technologies; see Section 7.

**Offers.** We model the interaction between a seller and the buyer(s) that she meets as a game in which the buyer(s) choose a mechanism and the seller chooses a message to send to each buyer she meets. A buyer’s mechanism is a function that maps the seller’s message into an offer, which specifies a quantity of numeraire to be exchanged for a certain fraction of the seller’s good.\(^\text{16}\) The seller’s message space can be arbitrarily large: it could include the quality of her good, whether or not she is in contact with the other buyer, the details of the other buyer’s mechanism, and any other (even not payoff-relevant) information. Importantly, we assume that mechanisms are exclusive, in the sense that a seller can choose to accept the offer generated by only one buyer’s mechanism, even when two offers are available.

In Appendix B, we apply insights from the delegation principle (Peters, 2001; Martimort and Stole, 2002) to show that, in our environment, it is sufficient to restrict attention to menu games where buyers offer a menu of two contracts.\(^\text{17}\) In particular, letting \(x\) denote the quantity of good to be exchanged for \(t\) units of numeraire, a buyer’s offer can be summarized by the menu \(\{(x_l, t_l), (x_h, t_h)\} \in ([0, 1] \times \mathbb{R}_+)^2\), where \((x_i, t_i)\) is the contract intended for a seller of type \(i \in \{l, h\}\).

\(^{15}\)Given the relationship in (3), it turns out that varying \(p\) or \(\pi\) is equivalent for all of our results below. We choose \(\pi\) because it simplifies some of the equations.

\(^{16}\)Note that the mechanisms we consider are assumed to be deterministic, but otherwise unrestricted. Stochastic mechanisms, or lotteries over menus, present considerable technical challenges and raise other conceptual issues that are, in our view, tangential to the paper’s main message.

\(^{17}\)To be more precise, we show that the (distribution of) equilibrium allocations in any game where buyers offer the general mechanisms described above coincide with those in another game in which buyers only offer a menu of two contracts.
**Payoffs.** A seller who owns a quality $i$ good and accepts a contract $(x, t)$ receives a payoff

$$ t + (1 - x)c_i $$

while a buyer who acquires a quality $i$ good at terms $(x, t)$ receives a payoff

$$ -t + xv_i. $$

Meanwhile, a seller with a quality $i$ good who does not trade receives a payoff $c_i$, while a buyer who does not trade receives zero payoff.

**Strategies and Definition of Equilibrium.** Let $z_i = (x_i, t_i)$ denote the contract that is intended for a seller of type $i \in \{l, h\}$, and let $z = (z_l, z_h)$. A buyer’s strategy, then, is a distribution across menus, $\Phi \in \Delta([0, 1] \times \mathbb{R}_+)^2$. A seller’s strategy is much simpler: given the available menus, a seller should choose the menu with the contract that maximizes her payoffs or mix between menus if she is indifferent. Of course, conditional on a menu, the seller chooses the contract that maximizes her payoffs. In what follows, we will take the seller’s optimal behavior as given.

A symmetric equilibrium is thus a distribution $\Phi^*(z)$ such that:

1. **Incentive compatibility:** for almost all $z = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^*(z)$,

   $$ t_l + c_l(1 - x_l) \geq t_h + c_l(1 - x_h) $$

   $$ t_h + c_h(1 - x_h) \geq t_l + c_h(1 - x_l). $$

2. **Buyer’s optimize:** for almost all $z = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^*(z)$,

   $$ z \in \arg \max_z \sum_{i \in \{l, h\}} \mu_i(v_i x_i - t_i) \left[ 1 - \pi + \pi \int_{z'} \chi_i(z, z') \Phi^*(dz') \right], $$

   where

   $$ \chi_i(z, z') = \begin{cases} 
   0 & \text{if } t_i + c_i(1 - x_i) < t'_i + c_i(1 - x'_i), \\
   \frac{1}{2} & \text{if } t_i + c_i(1 - x_i) = t'_i + c_i(1 - x'_i), \\
   1 & \text{if } t_i + c_i(1 - x_i) > t'_i + c_i(1 - x'_i). 
   \end{cases} $$

The function $\chi_i$ reflects the seller’s optimal choice. We have assumed that if the seller is indifferent between menus, then she chooses among menus with equal probability. Within a given menu, we have assumed that sellers do not randomize; for any incentive compatible contract, sellers choose the contract intended for their type, as in most of the mechanism design literature (see, e.g., Myerson (1985a), Dasgupta et al. (1979)).
3 Properties of Equilibria

Characterizing the equilibrium described above requires solving for a distribution over four-dimensional menus. In this section, we establish a series of results that reduce the dimensionality of the equilibrium characterization. First, we show that each menu offered by a buyer can be summarized by the indirect utilities that it delivers to each type of seller, so that equilibrium strategies can in fact be defined by a joint distribution over two-dimensional objects, i.e., pairs of indirect utilities. Then, we establish that the marginal distributions of offers intended for each type of seller are well-behaved, i.e., that they have fully connected support and no mass points. Finally, we establish that there is a very precise link between the two contracts offered by any buyer, which imposes even more structure on the joint distribution of offers. In particular, we show that any two menus that are offered in equilibrium are ranked in exactly the same way by both low- and high-type sellers; that is, one menu is strictly preferred by a low-type seller if and only if it is also preferred by a high-type seller. This property of equilibria, which we call “strictly rank-preserving,” simplifies the characterization even more, as the marginal distribution of offers for high-quality sellers can be expressed as a strictly monotonic transformation of the marginal distribution of offers for low-quality sellers.

3.1 Utility Representation

As a first step, we establish two results that imply any menu can be summarized by two numbers, \((u_l, u_h)\), where

\[ u_i = t_i + c_i(1 - x_i) \]  

(8)

denotes the utility received by a type \(i \in \{l, h\}\) seller from accepting a contract \(z_i\).

Lemma 1. In any equilibrium, for almost all \(z \in \text{supp}(\Phi^*)\), it must be that \(x_l = 1\) and \(t_l = t_h + c_l(1 - x_h)\).

In words, Lemma 1 states that all equilibrium menus require that low-quality sellers trade their entire endowment, and that their incentive compatibility constraint always binds. This is reminiscent of the “no-distortion-at-the-top” result in the taxation literature, or that of full insurance for the high-risk agents in Rothschild and Stiglitz (1976).

Corollary 1. In equilibrium, any menu of contracts \(\{(x_l, t_l), (x_h, t_h)\} \in \left([0, 1] \times \mathbb{R}_+\right)^2\) can be summarized by a pair \((u_l, u_h)\) with \(x_l = 1\), \(t_l = u_l\),

\[ x_h = 1 - \frac{u_h - u_l}{c_h - c_l}, \quad \text{and} \]

(9)

\[ t_h = \frac{u_l c_h - u_h c_l}{c_h - c_l}. \]

(10)
Notice that, since $0 \leq x_h \leq 1$, feasibility requires that the pair $(u_l, u_h)$ satisfies
\[ c_h - c_l \geq u_h - u_l \geq 0. \] (11)

In what follows, we will often refer to the requirement $u_h \geq u_l$ as a “monotonicity constraint.” Note that, when this constraint binds, Corollary 1 implies that $x_h = 1$ and $t_h = t_l$.

3.2 Recasting the Buyer’s Problem and Equilibrium

**Buyer’s Problem.** Lemma 1 and Corollary 1 allow us to recast the problem of a buyer as choosing a menu of indirect utilities, $(u_l, u_h)$, taking as given the distribution of indirect utilities offered by the other buyer. For any menu $(u_l, u_h)$, a buyer must infer the probability that the menu will be accepted by a type $i \in \{l, h\}$ seller. In order to calculate these probabilities, let us define the marginal distributions

\[ F_l(u_l) = \int_{z_l}^{u_l} 1 \left[ t_l' + c_l \left( 1 - x_l' \right) \leq u_l \right] \Phi \left( dz_l' \right) \]

for $i \in \{l, h\}$. In words, $F_l(u_l)$ and $F_h(u_h)$ are the probability distributions of indirect utilities arising from each buyer’s mixed strategy. When these distributions are continuous and have no mass points, the probability that a contract intended for a type $i$ seller is accepted is simply $1 - \pi + \pi F_l(u_l)$, i.e., the probability that the seller is captive plus the probability that he is noncaptive but receives another offer less than $u_l$. However, if $F_l(\cdot)$ has a mass point at $u_l$, then the fraction of noncaptive sellers of type $i$ attracted to a contract with value $u_l$ is given by $\tilde{F}_l(u_l) = \frac{1}{2} F_l(u_l) + \frac{1}{2} F_l(u_l)$, where $F_l(u_l) = \lim_{u_l \to u_l} F_l(u)$ is the left limit of $F_l$ at $u_l$. Given $\tilde{F}_l(\cdot)$, a buyer solves

\[
\begin{aligned}
\max_{u_l \geq c_l, u_h \geq c_h} & \quad \mu_l \left( 1 - \pi + \pi \tilde{F}_l(u_l) \right) \Pi_l(u_l, u_h) + \mu_h \left( 1 - \pi + \pi \tilde{F}_h(u_h) \right) \Pi_h(u_l, u_h) \\
\text{s. t.} & \quad c_h - c_l \geq u_h - u_l \geq 0,
\end{aligned}
\] (12)

with

\[
\begin{aligned}
\Pi_l(u_l, u_h) & \equiv v_l x_l - t_l = v_l - u_l \\
\Pi_h(u_l, u_h) & \equiv v_h x_h - t_h = v_h - u_h - \frac{v_h - c_l}{c_h - c_l} + u_l \frac{v_h - c_l}{c_h - c_l}
\end{aligned}
\] (13)

In words, $\Pi_l(u_l, u_h)$ is the buyer’s payoff conditional on the offer $u_l$ being accepted by a type $i$ seller. We refer to the objective in (12) as $\Pi(u_l, u_h)$.

Before proceeding, note that $\Pi_h(u_l, u_h)$ is increasing in $u_l$: by offering more utility to low-quality sellers, the buyer relaxes the incentive constraint and can earn more profits when he trades with high-quality sellers. As a result, one can easily show that the profit function $\Pi(u_l, u_h)$ is (at least) weakly supermodular. This property will be important in several of the results we establish below.
Equilibrium. Using the optimization problem described above, we can redefine the equilibrium in terms of the distributions of indirect utilities. In particular, for each \( u_l \), let

\[
U_h(u_l) = \arg \max_{u'_h \geq c_h} \Pi(u_l, u'_h) \\
\text{s.t. } c_h - c_l \geq u'_h - u_l \geq 0.
\]

The equilibrium can then be described by the marginal distributions \( \{F_i(u_i)\}_{i \in \{l, h\}} \) together with the requirement that a joint distribution function must exist. In other words, a probability measure \( \Phi \) over the set of feasible \((u_l, u_h)\)'s must exist such that, for each \( u_l > u'_l \) and \( u_h > u'_h \)

\[
1 = \Phi (\{ (\hat{u}_l, \hat{u}_h) ; \hat{u}_h \in U_h(\hat{u}_l) , \hat{u}_l \in [c_l, v_h] \})
\]

\[
F_l^- (u_l) - F_l (u'_l) = \Phi (\{ (\hat{u}_l, \hat{u}_h) ; \hat{u}_h \in U_h(\hat{u}_l) , \hat{u}_l \in (u'_l, u_l) \}), \tag{16}
\]

\[
F_h^- (u_h) - F_h (u'_h) = \Phi (\{ (\hat{u}_l, \hat{u}_h) ; \hat{u}_h \in U_h(\hat{u}_l) , \hat{u}_h \in (u'_h, u_h) \}). \tag{17}
\]

Note that this definition of equilibrium imposes two different requirements. The first is that buyers behave optimally: for each \( u_l \), the joint probability measure puts a positive weight only on \( u_h \in U_h(u_l) \).

The second is aggregate consistency: the fact that \( F_l \) and \( F_h \) are marginal distributions associated with a joint measure of menus.

3.3 Basic Properties of Equilibrium Distributions

In this section, we establish that, in equilibrium, the distributions \( F_l(u_l) \) and \( F_h(u_h) \) are continuous and have connected support, i.e., there are neither mass points nor gaps in either distribution.

**Proposition 1.** The marginal distributions \( F_l \) and \( F_h \) have connected support. They are also continuous, with the possible exception of a mass point in \( F_l \) at \( v_l \).

As in Burdett and Judd (1983), the proof of Proposition 1 rules out gaps and mass points in the distribution by constructing profitable deviations. A complication that arises in our model, which does not arise in Burdett and Judd (1983), is that payoffs are interdependent, e.g., a change in the utility offered to low-quality sellers changes the contract—and hence the profits—that a buyer receives from high-quality sellers. We prove the properties of \( F_l \) and \( F_h \) described in Proposition 1 sequentially: we first show that \( F_h \) is continuous and strictly increasing, and then apply an inductive argument to prove that \( F_l \) has connected support and is continuous, with a possible exception at the lower bound of the support. An important step in the induction argument, which we later use more generally, is to show that the objective function \( \Pi(u_l, u_h) \) is strictly supermodular. We state this here as a lemma.
Lemma 2. The profit function is strictly supermodular, i.e.,

$$\Pi(u_{l1}, u_{h1}) + \Pi(u_{l2}, u_{h2}) \geq \Pi(u_{l1}, u_{h1}) + \Pi(u_{l1}, u_{h2}), \quad \forall u_{l1} \geq u_{l2}, \quad i \in \{l, h\}$$

with strict inequality when $u_{l1} > u_{l2}, \quad i \in \{l, h\}$.

As noted above, the supermodularity of the buyer’s profit function reflects a basic complementarity between the indirect utilities offered to low- and high-quality sellers. An important implication of this result is that the correspondence $U_h(u_l)$ is weakly increasing. We use this property to construct deviations to rule out gaps and mass points in the distribution $F_l$ almost everywhere in its support; later, in Section 4, we show that these mass points only occur in a knife-edge case. Hence, generically, the marginal distribution $F_l$ has connected support and no mass points everywhere in its support.

3.4 Strict rank-preserving

In this section, we establish that every equilibrium has the property that the menus being offered are strictly rank-preserving—that is, low- and high-quality sellers share the same ranking over the set of menus offered in equilibrium—with the possible exception of the knife-edge case discussed above. We prove this result by showing that the mapping between a buyer’s optimal offer to low- and high-quality sellers, $U_h(u_l)$, is a well-defined, strictly increasing function. We start with the following definition.

Definition 1. For any subset $U_l$ of $\text{Supp}(F_l)$, an equilibrium is strictly rank-preserving over $U_l$ if the correspondence $U_h(u_l)$ is a strictly increasing function of $u_l$ for all $u_l \in U_l$. An equilibrium is strictly rank-preserving if it is strictly rank-preserving over $\text{Supp}(F_l)$.

Equivalently, an equilibrium is strictly rank-preserving when, for any two points in the equilibrium support $(u_l, u_h)$ and $(u'_l, u'_h)$, $u_l > u'_l$ if and only if $u_h > u'_h$. Given this terminology, we can now establish one of our key results.

Theorem 1. All equilibria are strictly rank-preserving over the set $\text{Supp}(F_l) \setminus \{v_l\}$.

Theorem 1 follows from the facts established above. In particular, the strict supermodularity of $\Pi(u_l, u_h)$ implies that $U_h(u_l)$ is a weakly increasing correspondence. However, since $F_l(\cdot)$ and $F_h(\cdot)$ are strictly increasing and continuous, we show that $U_h(u_l)$ can neither be multi-valued nor have flats. Intuitively, if there exists a $u_l > u_l$ and $u'_h > u_h$ such that $u_h, u'_h \in U_h(u_l)$, then the supermodularity of $\Pi(u_l, u_h)$ implies that $[u_h, u'_h] \subset U_h(u_l)$. Since $F_h(\cdot)$ has connected support, if $U_h$ were a correspondence for some $u_l$, then this would imply that $F_l(\cdot)$ must have a mass point at $u_l$, which
contradicts Proposition 1. Similarly, if there exists \( u_h \) and \( u'_h > u_l \) offered in equilibrium such that \( U_h(u'_h) = U_h(u_l) = u_h \), then \( F_h \) would feature a mass point, in contradiction with Proposition 1. Hence, \( U_h(u_l) \) must be a strictly increasing function for all \( u_l > u_l \).

Notice that, if \( F_l(\cdot) \) is continuous everywhere, then every menu offered in equilibrium is accepted by exactly the same fraction of low- and high-quality noncaptive sellers. We state this result in the following Corollary to Theorem 1.

**Corollary 2.** If \( F_l \) and \( F_h \) are continuous, then \( F_h(U_h(u_l)) = F_l(u_l) \).

Taken together, Theorem 1 and Corollary 2 simplify the construction of an equilibrium, which we undertake in the next section. Specifically, when an equilibrium exists in which the marginal distributions \( F_l \) and \( F_h \) are continuous, then the equilibrium can be described compactly by the marginal distribution \( F_l \) and the policy function \( U_h(u_l) \).

### 4 Construction of Equilibrium

In this section, we use the properties established above to help construct equilibria. Then, we show that the equilibrium we construct is unique. In this sense, we characterize the entire set of equilibrium outcomes in our model.

#### 4.1 Special Cases: Monopsony and Perfect Competition

To fix ideas, we first characterize equilibria in the well-known special cases of \( \pi = 0 \) and \( \pi = 1 \), i.e., when sellers face a monopsonist and when they face two buyers in Bertrand competition, respectively. As we will see, several features of the equilibrium in these two extreme cases guide our construction of equilibria for the general case of \( \pi \in (0,1) \).

**Monopsony.** When each seller meets with at most one buyer, the buyers solve

\[
\max_{(u_l, u_h)} \mu_l(v_l - u_l) + \mu_h \left[ v_h - u_h \frac{v_h - c_l}{c_h - c_l} + u_l \frac{v_h - c_h}{c_h - c_l} \right],
\]

subject to the monotonicity and feasibility constraints in (13). The solution to this problem, summarized in Lemma 3 below, is standard and, hence, we omit the proof.

**Lemma 3.** Suppose \( \pi = 0 \), and let

\[
\phi_l = 1 - \frac{\mu_h}{\mu_l} \left( \frac{v_h - c_h}{c_h - c_l} \right). \tag{18}
\]
If $\phi_l > 0$, then the unique equilibrium has $u_l = c_l$ with $x_l = 1$ and $u_h = c_h$ with $x_h = 0$; if $\phi_l < 0$, then $u_l = u_h = c_h$ with $x_l = x_h = 1$; and if $\phi_l = 0$, then $u_l \in [c_l, c_h]$ with $x_l = 1$ and $u_h = c_h$ with $x_h \in [0, 1]$.

The parameter $\phi_l$ is a summary statistic for the adverse selection problem: it represents the net marginal cost (to the buyer) of delivering an additional unit of utility to a low-quality seller. It is strictly less than 1 because the direct cost of an additional unit of transfer to a low-quality seller is partially offset by the indirect benefit of relaxing this seller’s incentive constraint, which allows the buyer to trade more with a high-quality seller. This indirect benefit is captured by the second term on the right-hand side: when this term is large, $\phi_l$ is small, the cost of trading with high-quality sellers is low, and adverse selection is mild. Conversely, when this term is small, $\phi_l$ is large, it is costly to trade with high-quality sellers, and therefore adverse selection is relatively severe. According to this measure, adverse selection is thus severe when the relative fraction of high-quality sellers, $\mu_h/\mu_l$, is small; the gains from trading with high-quality sellers, $v_h - c_h$, are relatively small; and/or the information rents associated with separating high- and low-quality sellers, $c_h - c_l$, are large.

When $\phi_l > 0$, the net cost to a buyer of increasing $u_l$ is positive, so she sets $u_l$ as low as possible, i.e., $u_l = c_l$. This implies that the high-quality seller is entirely shut out, i.e., $x_h = 0$. Otherwise, when $\phi_l < 0$, increasing $u_l$ yields a net benefit to the buyer. As a result, a buyer raises $u_l$ until the monotonicity constraint in (13) binds, i.e., she pools high- and low-quality sellers, offering $u_h = u_l = c_h$.

Before proceeding to the perfectly competitive case, we highlight two features of the equilibrium under monopsony. First, buyers offer separating menus ($u_h > u_l$) when $\phi_l$ is positive and pooling menus ($u_h = u_l$) when $\phi_l$ is negative. Second, they make non-negative payoffs on both types when $\phi_l > 0$, but lose money on low-quality sellers when $\phi_l < 0$. In other words, the equilibrium features cross-subsidization when $\phi_l$ is negative, but not when $\phi_l$ is positive.

**Bertrand Competition.** When competition is perfect, i.e., when $\pi = 1$, our setup becomes the same as that in Rosenthal and Weiss (1984), and similar to that of Rothschild and Stiglitz (1976). In this case, when $\phi_l \geq 0$, the unique equilibrium is in pure strategies, with buyers offering the standard “least-cost separating” contract; type l sellers earn $u_l = v_l$ and type h sellers trade a fraction of their endowment at a unit price of $v_h$, such that the incentive constraint of the low-quality seller binds. However, when $\phi_l < 0$, there is no pure strategy equilibrium.\(^\text{18}\) In this case, an equilibrium in mixed strategies emerges,

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\(^\text{18}\)All buyers offering the least-cost separating contract cannot be an equilibrium, as a pooling offer attracts both types and yields positive profits to the buyer. All buyers offering pooling cannot be an equilibrium either, since it is vulnerable to a cream-skimming deviation, wherein a competing buyer can draw away the high-quality seller by offering a contract with $x < 1$ but at a higher price.
as in Rosenthal and Weiss (1984) and Dasgupta and Maskin (1986). Each buyer mixes over menus, all of which involve negative profits from low-quality sellers, offset exactly by positive profits from high-quality sellers, leading to zero profits. The marginal distribution \( F_t(\cdot) \) is such that profitable deviations are ruled out. The following lemma summarizes these results.

**Lemma 4.** When \( \pi = 1 \), the unique equilibrium is as follows: (i) if \( \phi_1 \geq 0 \), then \( u_1 = v_1 \) with \( x_1 = 1 \) and \( u_h = \frac{v_h(c_h-c_l)+v_l(v_h-c_l)}{v_h-c_l} \) with \( x_h = \frac{v_l-c_l}{v_h-c_l} \); (ii) if \( \phi_1 < 0 \), then the symmetric equilibrium is described by the distribution

\[
F_1(u_t) = \left( \frac{u_t - v_1}{\mu_h (v_h - v_1)} \right)^{-\phi_1} \tag{19}
\]

with \( \text{Supp}(F_1) = [v_1, \bar{v}] \) and \( F_h(u_h) = F_1(U_h(u_1)) \), where \( \bar{v} = \mu_h v_h + \mu_1 v_1 \) and \( U_h(u_1) \) satisfies

\[
\mu_h \Pi_h(u_l, U_h(u_1)) + \mu_l \Pi_l(u_l, U_h(u_1)) = 0. \tag{20}
\]

As with \( \pi = 0 \), equilibrium when \( \pi = 1 \) features no cross-subsidization when \( \phi_1 \geq 0 \) and cross-subsidization when \( \phi_1 < 0 \). However, unlike the case with \( \pi = 0 \), equilibrium with \( \pi = 1 \) features separating contracts for all values of \( \phi_1 \). These properties guide our construction of equilibria in the next section, when we study the general case of \( \pi \in (0, 1) \).

### 4.2 General Case: Imperfect Competition

We now describe how to construct equilibria when \( \pi \in (0, 1) \). Recall that an equilibrium is summarized by a distribution \( F_1(u_t) \) and a strictly increasing function \( U_h(u_1) \). A key determinant of the structure of equilibrium menus is whether the monotonicity constraint in (13) is binding. When it is slack, the local optimality (or first-order) condition for \( u_t \), along with the strict rank-preserving condition that relates \( F_h(U_h(u_1)) = F_1(u_1) \) together characterize the equilibrium distribution \( F_1(u_t) \). The function \( U_h(u_1) \) then follows from the requirement that all menus \((u_1, U_h(u_1))\) must yield the buyer equal profits. When the monotonicity constraint is binding, the policy function is, by definition, \( U_h(u_1) = u_1 \).

Our analysis of \( \pi = 0 \) and \( \pi = 1 \) points to the importance of \( \phi_1 \). Recall that when \( \phi_1 > 0 \), the monotonicity constraint was always slack. When \( \phi_1 < 0 \), on the other hand, the monotonicity constraint was binding only when \( \pi = 0 \) and slack at \( \pi = 1 \). Guided by these results, we discuss our construction separately for the \( \phi_1 > 0 \) and the \( \phi_1 < 0 \) cases.

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19 Luz (2014) shows that the equilibrium is unique.

20 Of course, \( u_h = U_h(u_1) \) must be locally optimal as well, but this condition is implied by the joint requirements on \( u_1 \) and \( U_h(u_1) \) described above.

21 The equilibrium when \( \phi_1 = 0 \) has a slightly different structure and, for the sake of brevity, we relegate analysis of this knife-edge case to Appendix C.
**Case 1:** $\phi_l > 0$. Given the analysis of $\pi = 0$ and $\pi = 1$, we conjecture that, for any $\pi \in (0, 1)$, the monotonicity constraint is slack, i.e., that $U_h(u_l) > u_l$ for all $u_l \in \text{Supp}(F_l)$. Proposition 2 establishes that this is indeed the case.

**Proposition 2.** For any $\pi \in (0, 1)$ and $\phi_l > 0$, there exists an equilibrium where $F_l$ and $U_h$ satisfy the following properties:

1. $F_l$ solves the differential equation
   \[
   \frac{\pi f_l(u_l)}{1 - \pi + \pi f_l(u_l)} (v_l - u_l) = \phi_l, \tag{21}
   \]
   with the boundary condition $F_l(c_l) = 0$.

2. $U_h(u_l) > u_l$ and satisfies the equal profit condition:
   \[
   (1 - \pi + \pi f_l(u_l)) [\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l (v_l - u_l)] = \mu_l (1 - \pi) (v_l - c_l). \tag{22}
   \]

Equation (21) is derived by taking the first-order condition of (12) with respect to $u_l$—holding $u_h$ fixed—and then imposing the strict rank-preserving property.\(^{22}\) This necessary condition is familiar from basic production theory. The left-hand side is the marginal benefit to the buyer of increasing $u_l$, i.e., the product of the semi-elasticity of demand and the profit per trade. The right-hand side, $\phi_l$, represents the marginal cost of increasing the utility of the low-quality seller, taking into account the fact that increasing $u_l$ relaxes the incentive constraint.\(^{23}\) Note that, even though (21) ensures that local deviations by a buyer from an equilibrium menu are not profitable, completing the proof requires ensuring that there are no profitable global deviations as well; we establish that this is true in Appendix A.2.1.

The boundary condition requires that the lowest utility offered to the low-quality seller is $c_l$. From (22), and the fact that $F_l(c_l) = 0$, we find $U_h(c_l) = c_h$, so that the worst menu offered in equilibrium coincides with the monopsony outcome. Intuitively, if the worst menu offers more utility to low-quality sellers than $c_l$, the buyer could profit by decreasing $u_l$ and $u_h$; the gains associated with trading at better terms with the low types would exceed the losses associated from trading less quantity with high types, precisely because $\phi_l > 0$. Given that $u_l = c_l$, if the worst equilibrium menu offers more utility to high-quality sellers than $c_h$, then a buyer offering this menu could profit by decreasing $u_h$; his payoff from trading with high types would increase without changing the payoffs from trading with low types.

\(^{22}\)As we discuss in the proof of Proposition 2, this first-order condition requires three assumptions: that $u_h > u_l$ for all menus; that there is no mass point at the lower bound of the support of $F_l(u_l)$; and that the implied quantity traded by the high-quality seller is interior in all trades, i.e., $0 < x_h = (u_h - u_l)/(c_h - c_l) < 1$, except possibly at the boundary of the support of $F_l$. All of these assumptions are confirmed in equilibrium.

\(^{23}\)It is straightforward to derive a closed-form solution for $F_l(u_l)$ from (21); see equation (48) in the Appendix.
The final equilibrium object, $U_h(u_l)$, is characterized by the equal profit condition: the left side of (22) defines the buyer’s payoff from the menu $(u_l, U_h(u_l))$, while the right side is the profit earned from the worst contract offered in equilibrium. Figure 1 plots the two equilibrium functions in this region.

Figure 1: Equilibrium for $\pi \in (0, 1)$, $\phi_l > 0$. The left panel plots the CDF $F_l(u_l)$ and the right panel plots the mapping $U_h(u_l)$.

Notice from (21) that, since $\phi_l > 0$, our equilibrium has $v_l > u_l$ for all menus in equilibrium, so that buyers earn strictly positive profits from trading with low-quality sellers. It is straightforward to show that buyers also earn strictly positive profits from trading with high-quality sellers. Hence, in this region, the equilibrium features no cross-subsidization, as was the case for $\pi = 0$ and $\pi = 1$. Finally, it is also worth noting that the equilibrium distribution of offers converges to the limiting cases as $\pi$ converges to both 0 and 1; in the former case, the distribution converges to a mass point at the monopsony outcome, while in the latter case, the distribution converges to a mass point at the least-cost separating outcome.

**Case 2: $\phi_l < 0$.** In this region of the parameter space, the equilibrium features a pooling menu when $\pi = 0$ and a distribution of separating menus when $\pi = 1$. This leads us to conjecture that the equilibrium for $\pi \in (0, 1)$ can feature pooling, separating, or a mixture of the two, depending on the value of $\pi$. The following lemma formalizes this conjecture and shows the existence of a threshold utility for the offer to low-quality sellers, such that all offers with $u_l$ below this threshold are pooling menus, while all offers above the threshold are separating menus.\(^{24}\) Depending on whether this threshold lies at the lower bound, the upper bound, or in the interior of the support of $F_l(u_l)$, there are three possible cases, respectively: all equilibrium offers are separating menus, all are pooling menus, or there is a mixture with some pooling menus (offering relatively low utility to the seller) and some separating

\(^{24}\)At this point, it may seem arbitrary to conjecture that pooling occurs at the bottom of the distribution and separation at the top. As we will discuss later in the text, the reason this is ultimately true is that the cream-skimming deviation—which makes the pooling offer suboptimal—becomes more attractive as the indirect utility being offered increases.
menus (offering higher utility). Later, in Proposition 4, we provide conditions on $\phi_l$ and $\pi$ under which each case obtains.

**Proposition 3.** For any $\pi \in (0, 1)$ and $\phi_l < 0$, there exists an equilibrium where $F_l$ and $U_h$ satisfy the following properties:

1. There exists a threshold $\hat{u}_l$ such that, for any $u_l$ in the interior of $\text{Supp}(F_l)$:
   
   - (a) if $u_l \leq \hat{u}_l$, $U_h(u_l) = u_l$ and $F_l$ satisfies
     \[
     \frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} (\mu_h v_h + \mu_l v_l - u_l) = 1, \tag{23}
     \]
   - (b) if $u_l > \hat{u}_l$, $U_h(u_l) > u_l$ and $F_l$ satisfies (21).

2. $U_h(\bar{u}_l) = c_h$ and $U_h(\underline{u}_l) = \bar{u}_l$.

To understand the first set of (necessary) conditions in Proposition 3, consider the region where the buyers offer pooling menus. Here, buyers trade off profit per trade against the probability of trade, with no interaction between offers and incentive constraints. As a result, the equilibrium in this pooling region behaves as in the canonical Burdett and Judd (1983) single-quality model, with the buyer’s payoff equal to the average value $\mu_h v_h + (1 - \mu_h) v_l$. This yields (23). In the region where buyers offer separating menus, $F_l(u_l)$ is characterized by the local optimality condition (21), exactly as in the $\phi_l > 0$ case. Recall from our discussion that this differential equation accounts explicitly for the effect of an offer $u_l$ on the seller’s incentive constraint. In this region, $U_h(u_l)$ is determined by the equal profit condition.

The second part of the result describes boundary conditions for the worst and best menus offered in equilibrium. The first condition requires that the worst menu yields utility $c_h$ to high-quality sellers. To see why, suppose the worst menu is a pooling menu with $U_h(\bar{u}_l) = \bar{u}_l > c_h$. Then, lowering both $u_h$ and $u_l$ leads to strictly higher profits. If the worst menu is separating with $U_h(\bar{u}_l) > c_h$, then a downward deviation in only $u_h$ is feasible and strictly increases profits. The second condition requires that the best menu offered in equilibrium is a pooling menu. Intuitively, if the best menu offered in equilibrium were a separating menu, then $x_h < 1$. This cannot be optimal when $\phi_l < 0$: the buyer can trade more with the high-quality seller by increasing the utility offered to low-quality sellers. Since this is already the best menu in equilibrium, this deviation has no impact on the number of sellers the buyer attracts but yields strictly higher profits.

Given these properties, we now establish two critical values—$\phi_1(\pi)$ and $\phi_2(\pi)$, with $\phi_2(\pi) < \phi_1(\pi) < 0$—that determine which of the three cases described above emerge in equilibrium. When $\phi_l < \phi_2(\pi)$,
the threshold $\hat{u}_1 = \bar{u}_1$ and there is an all pooling equilibrium. When $\phi_1 > \phi_1(\pi)$, the monotonicity constraint is slack almost everywhere, so that $\hat{u}_1 = \underline{u}_1$, and the equilibrium features all separating menus. Finally, if $\phi_1$ lies between these two critical values, we have a mixed equilibrium, with an intermediate threshold $\hat{u}_1 \in (\underline{u}_1, \overline{u}_1)$. Figure 2 illustrates $U_{h}(u_1)$ for all three possibilities.

**Proposition 4.** For any $\pi \in (0, 1)$, there exist two cutoffs $\phi_2(\pi) < \phi_1(\pi) < 0$ such that an all pooling equilibrium exists for all $\phi_1 \leq \phi_2(\pi)$, a mixed equilibrium exists for all $\phi_1 \in (\phi_2(\pi), \phi_1(\pi))$, and an all separating equilibrium exists for all $\phi_1 \in (\phi_1(\pi), 0)$.

Intuitively, for a pooling menu $(\underline{u}_1, \overline{u}_1)$ to be offered in equilibrium, the cream-skimming deviation $(\underline{u}_1 - \varepsilon, \overline{u}_1)$ for some $\varepsilon > 0$ cannot yield strictly higher profits. To see how incentives to cream-skim vary with $\phi_1$ and $\pi$, notice that there are two sources of higher profits from the menu $(\underline{u}_1 - \varepsilon, \overline{u}_1)$, relative to the candidate pooling menu. First, it decreases the loss conditional on trading with a low-quality seller. Second, it reduces the probability of trading with a noncaptive low-quality seller; since the buyer loses money on these sellers, this reduction in trading probability raises profits. The cost of cream-skimming is that the buyer earns lower profits on high-quality sellers. Therefore, incentives to cream-skim are weak—and thus pooling is easier to sustain—when high-quality sellers are relatively abundant ($\phi_1$ very negative) and/or there are relatively few noncaptive sellers ($\pi$ is small).

The higher the level of utility being offered in a pooling menu, the more vulnerable it is to cream-skimming. Hence, if such a deviation is profitable at the lowest candidate value, $c_h$, then pooling cannot be sustained at all: this is the condition that determines the cutoff $\phi_1(\pi)$. Similarly, the cutoff $\phi_2(\pi)$ defines the boundary at which cream-skimming is not profitable even at the best pooling menu, $\overline{u}_1$. We derive these thresholds formally and provide a full equilibrium characterization in Appendix A.2.2.

Notice that, in all three cases, $\underline{u}_1 > v_1$ (since $\underline{u}_1 \geq c_h > v_1$) so that buyers always suffer losses.
when trading with low-quality sellers. Hence, as in the extreme cases of \( \pi = 0 \) and \( \pi = 1 \), there is cross-subsidization in every equilibrium when \( \phi_1 < 0 \). Finally, as in the case of \( \phi_1 > 0 \), the equilibrium distribution converges to the limiting cases as \( \pi \) converges to both 0 and 1.

Figure 3 summarizes the various types of equilibria and the regions in which each one obtains. The x- and y-axes represent the intensity of competition and severity of adverse selection, respectively. Recall that the latter is summarized by \( \phi_1 \), which is a function of \( \mu_h \), the fraction of high-quality goods, as well as the valuations \( v_h, c_h, c_l \). For concreteness, we use \( \mu_h \) to vary \( \phi_1 \) on the y-axis—a higher fraction of high-quality goods implies a lower \( \phi_1 \) and, therefore, milder adverse selection.\(^{25}\)

![Figure 3: Equilibrium regions](image)

### 4.3 Uniqueness

In the previous section, we constructed equilibria for all \( \pi \in (0, 1) \) and \( \phi_1 \leq 1 \). In Theorem 2, below, we establish that these equilibria are unique. For intuition, we sketch the arguments here for \( \phi_1 \neq 0 \).\(^{26}\) First, we show that for all \( \phi_1 \neq 0 \), no equilibrium features a mass point, even at \( v_1 \). Next, when \( \phi_1 > 0 \), we prove that no equilibrium features pooling menus on a positive measure subset of \( \Gamma_1 \). In this case, since equilibria have no mass points and must be separating almost everywhere, the equilibrium we construct in Proposition 2 describes the unique equilibrium.

When \( \phi_1 < 0 \), we demonstrate uniqueness of the equilibrium with a threshold \( \hat{u}_1 \) in steps. First, we show that any equilibrium features pooling at the upper bound of the support of \( \Gamma_1 \). Second, we prove that any equilibrium features at most one interval of pooling menus followed by at most one interval

\(^{25}\)The boundaries are also redefined accordingly: \( \mu_h < \mu_0 \) if and only if \( \phi_1 \geq 0 \) and \( \mu_h < \mu_j(\pi) \) if and only if \( \phi_1 \geq \phi_j(\pi) \) for \( j \in \{1, 2\} \).

\(^{26}\)In Appendix C, we also prove uniqueness for the knife-edge case of \( \phi_1 = 0 \).
of separating menus. Third, we prove that the equilibria characterized in Proposition 4 are mutually exclusive, so that equilibria without mass points are unique. Since no equilibrium features mass points when $\phi_l < 0$, these results establish the uniqueness of the equilibrium characterized in Proposition 4. We summarize these results in the following theorem.

**Theorem 2.** For any $\pi \in (0, 1)$ and $\phi_l \in \mathbb{R}$, there exists an equilibrium and it is unique.

Note that we obtain a unique equilibrium without any refinements or other restrictions on off-path behavior. This is because buyers’ payoffs are well-defined for any offer. In particular, since buyers are not capacity-constrained, the fraction of type $i \in \{l, h\}$ sellers who accept any offer $(u_l, u_h)$ is uniquely determined by the (exogenous) meeting technology and the (endogenous) distribution of offers $F_l$ and $F_h$.\(^{27}\)

### 4.4 Discussion

The equilibrium characterized above has a number of testable implications for transaction prices and quantities. The first set of predictions pertains to properties of equilibrium menus. We highlight three robust predictions. First, the strict rank-preserving property suggests a positive correlation between the contracts that buyers offer to different types of sellers: those buyers who make attractive offers to low-quality sellers will also make attractive offers to high-quality sellers. Hence, in equilibrium, buyers do not specialize in trading with a particular type of seller, but rather trade with equal frequency across all types. Second, whether buyers pool different types of sellers or separate them (using a menu of options) depends crucially on the severity of the two frictions.\(^{28}\) Pooling is more likely in markets where competition among buyers is relatively weak and adverse selection is relatively mild. Alternatively, separation is more likely when adverse selection is relatively severe—so that the information costs of trading with high-quality sellers are large relative to the benefits—and competition is relatively strong—so that the payoffs from cream-skimming are relatively high.\(^{29}\) Third, the theory also predicts that menus that are less attractive from the perspective of sellers are more likely to be pooling. In other words, those who are posting offers with relatively unattractive terms should be offering fewer options and should account for a smaller share of observed transactions.

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\(^{27}\)Refinements are often necessary in models with capacity-constrained buyers, when two types of sellers would like to accept an off-path offer, and the probability that each type is able to execute the trade is not pinned down.

\(^{28}\)This result stands in stark contrast to, e.g., Guerrieri et al. (2010). In that model, and many like it, the quantity traded with high-quality sellers is independent of the distribution of types in the market; trade with high-quality sellers is distorted even if the fraction of low-quality goods in the market is arbitrarily small.

\(^{29}\)Consistent with our findings, Decarolis and Guglielmo (2015) find evidence of greater cream-skimming by health insurance providers when the market is more competitive.
The second set of implications pertains to dispersion. Note that, in the region with separating menus, the model predicts dispersion within and across types. This is true both for quantities traded (coverage in an insurance context or loan size in a credit market context) as well as prices (premia or interest rates, respectively). The extent of dispersion—both the support and the standard deviation of the quantity/price distributions—is determined by the interaction of competition (measured by $\pi$) and adverse selection (measured by $\phi_1$). This joint dependence calls into question the practice of identifying imperfect competition or asymmetric information in isolation using cross-sectional dispersion. For example, a common empirical strategy to identify adverse selection is to test the correlation between the quantity an agent trades and her type, as measured by ex-post outcomes.\(^{30}\) In our equilibrium, there is a negative correlation between the seller’s quality and the quantity she sells, but the quantitative strength of this relationship is also a function of the market structure. As a result, using the relationship between quantity and type without accounting for the imperfect nature of competition is likely to yield misleading conclusions. A similar concern applies to the strategy of identifying search frictions from price dispersion.\(^{31}\) In markets where adverse selection is a concern, the extent of cross-sectional variation in terms of trade is also a function of selection-related parameters. Obtaining an accurate assessment of trading frictions in such settings thus requires controlling for the underlying distribution of types.

### 5 Increasing Competition and Reducing Information Asymmetries

Many markets in which adverse selection is a first-order concern are experiencing dramatic changes. Some of these changes are regulatory in nature; for example, as we describe in greater detail below, there are several recent policy initiatives to make health insurance markets and over-the-counter markets for financial securities more competitive and transparent. Other changes derive from technological improvements; for example, advances in credit scoring reduce information asymmetries in loan markets.

In this section, we use the framework developed above to examine the likely effects of these types of changes on economic activity. Our metric for economic activity is the utilitarian welfare function, which measures the expected gains from trade that are realized in equilibrium. We show that increasing competition or reducing information asymmetries can worsen the distortions from adverse selection—thereby decreasing the expected gains from trade—when markets are relatively competitive. As a result, initiatives to make these markets more competitive or transparent are only welfare-improving when both

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\(^{30}\)This technique for identifying adverse selection has been applied to a number of markets, following the seminal paper by Chiappori and Salanie (2000); recent examples include Ivashina (2009), Einav et al. (2010b), and Crawford et al. (2015).

\(^{31}\)Using dispersion in prices to help identify search frictions is standard practice in the IO literature; for a recent example, see Gavazza (2015).
frictions are relatively severe, i.e., when buyers have a lot of market power (i.e., when $\pi$ is low) and the adverse selection problem is relatively severe (i.e., when $\phi_l$ is high).

While these comparative statics are certainly informative, one may be concerned that they reflect an inefficiency in the particular game we postulate between buyers and sellers. At the end of this section, we derive a constrained efficient benchmark, taking the search and information frictions as given. We show that, in the region of the parameter space where $\phi_l > 0$, equilibrium gains from trade coincide with those in the constrained efficient benchmark. This suggests that the comparative statics results that we described above are not a consequence of the particular game we have modeled, but rather a more fundamental feature of markets with adverse selection and imperfect competition.

5.1 Utilitarian Welfare

As noted above, our metric for economic activity will be the objective of a utilitarian planner, defined as the expected gains from trade realized between buyers and sellers, or

$$W(\pi, \mu_h) = (1 - \mu_h)(v_l - c_l) + \mu_h \left\{ \frac{2 - 2\pi}{2 - \pi} \int [x_h(u_l)(v_h - c_h)] dF_l(u_l) + \frac{\pi}{2 - \pi} \int \mu_h [x_h(u_l)(v_h - c_h)] d\left(F_l(u_l)^2\right) \right\},$$

where, in a slight abuse of notation, we let

$$x_h(u_l) = 1 - \frac{\mu_h(u_l) - u_l}{c_h - c_l}. \quad (25)$$

The first term in (24) represents the gains from trade generated by low quality goods; since all sellers receive at least one offer and $x_l = 1$ in every trade, all low-quality goods are transferred to the buyer. The second term captures the expected gains from trade between buyers and captive high-quality sellers. In particular, from equation (3), we can write the measure of captive sellers as $1 - p = \frac{2 - 2\pi}{2 - \pi}$. A randomly selected captive high-quality seller transfers $x_h(u_l)$ to the buyer and consumes the remaining $1 - x_h(u_l)$ herself, where $u_l$ is drawn from $F_l(u_l)$. Finally, the last term in (24) captures the expected gains from trade between buyers and noncaptive high-quality sellers. A measure $p = \frac{\pi}{2 - \pi}$ of sellers are noncaptive and, since noncaptive sellers choose the maximum indirect utility among the two offers they receive, they trade an amount $x_h(u_l)$ where $u_l$ is drawn from $F_l(u_l)^2$. 
5.2 Increasing Competition

We first study the effects of increasing competition, which has been a common policy response to address perceived failures in markets for insurance, credit, and certain types of financial securities.\(^{32}\) We do so by examining the relationship between welfare and competition, as captured by \(\pi\). In Proposition 5, we establish that welfare is maximized at \(\pi = 0\) when the adverse selection problem is relatively mild. However, when the adverse selection problem is severe, we show that \(W\) is hump-shaped in \(\pi\); i.e., there is an interior level of competition that maximizes welfare in this region of the parameter space.

**Proposition 5.** If \(\phi_l \leq 0\), welfare is maximized at \(\pi = 0\). Otherwise, it is maximized at a \(\pi \in (0, 1]\).

The first result is straightforward. Since a monopsonist offers a pooling contract in this region of the parameter space, all gains from trade are realized. Competition only serves to increase incentives to cream-skim. When these incentives are sufficiently strong, equilibrium menus offer high-quality sellers a higher price but a lower quantity to trade in order to ensure that such a deviation is not profitable, causing a decline in welfare.

The second result—that welfare is maximized at an interior value of \(\pi\) when \(\phi_l > 0\)—is less obvious. To see the intuition for this result, first note that, as \(\pi\) increases, \(F_l(u_l)\) increases in the sense of first-order stochastic dominance: \(F_l(u_l)\) shifts to the right and \(\bar{u}_l\) increases. Intuitively, in equilibrium, more competition forces buyers to allocate more surplus to sellers. Second, and crucially, \(x_h(u_l)\) is hump-shaped in \(u_l\): it increases near the monopsony offer \(c_l\), and decreases when \(u_l\) is sufficiently close to the competitive offer, \(v_l\). When \(\pi\) is close to zero, \(\bar{u}_l\) is relatively small and the distribution of offers is clustered near the monopsony contract; a small increase in \(\pi\) causes a rightward shift in the density of offers to values of \(u_l\) associated with higher values of \(x_h\), increasing the gains from trade realized between buyers and high-quality sellers. In contrast, when \(\pi\) is close to 1, \(\bar{u}_l\) is close to \(v_l\), and the distribution of offers is clustered near the competitive contract; in this case, a small increase in \(\pi\) causes a shift toward values of \(u_l\) associated with lower values of \(x_h\).

Therefore, understanding why welfare is hump-shaped in \(\pi\) ultimately requires understanding why \(x_h(u_l)\) is hump-shaped in \(u_l\). Note that, ceteris paribus, an increase in \(u_l\) relaxes the type \(l\) seller’s

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\(^{32}\)For example, a recent report by the Congressional Budget Office (2014) argues for “fostering greater competition” in health insurance plans by developing “policies that would increase the average number of sponsors per region,” which would then “increase the likelihood that beneficiaries would select low-cost plans.” Similarly, the U.S. Treasury (2010) argued that the Consumer Financial Protection Bureau “will make consumer financial markets more transparent – and that’s good for everyone: The agency will give Americans [...] the tools they need to comparison shop for the best prices and the best loans, which will [...] increase competition and innovations that benefit borrowers.” A similar rationale underlies the Core Principles and Other Requirements for Swap Execution Facilities (Commodity Futures Trading Commission (2013)), issued under the Dodd-Frank Wall Street Reform and Consumer Protection Act, which requires that a swap facility sends a buyer’s request for price quotes to a minimum number of sellers before a trade can be executed.
incentive compatibility constraint, allowing buyers to raise $x_h$. In contrast, *ceteris paribus*, an increase in $u_h$ tightens the type l seller’s incentive compatibility constraint, requiring buyers to lower $x_h$. Thus, as offers to both types increase, the net effect on $x_h$ depends on which one rises faster—formally, whether $U_h'(u_l)$ is greater or less than 1. Figure 4 illustrates this relationship between the quantity traded with high types, $x_h$, and the rate at which $u_h$ and $u_l$ increase within the set of equilibrium menus being offered. The figure reveals that $u_l$ rises faster than $u_h$ for smaller values of $u_l$, so that $x_h$ is increasing in this region. However, as $u_l$ nears $v_l$, $u_h$ rises faster and thus $x_h$ is decreasing in this region.

To explain the hump-shape of welfare, then, we need to understand why $U_h'(u_l) < 1$ for low levels of $u_l$ and $U_h'(u_l) > 1$ for high levels of $u_l$. While this slope is a complicated equilibrium object, determined by the interaction of an individual buyer’s optimal strategy and the equilibrium distribution of offers, the basic intuition can be understood through two opposing forces. First, it is cheaper for buyers to provide utility to the low type (relative to the high type) because doing so has the additional benefit of relaxing the incentive constraints; we call this the “incentive effect” and this force tends to reduce the slope, $U_h'(u_l)$. Second, as $u_l$ rises, buyers have more incentive to attract type h sellers, relative to type l sellers; formally, one can show that $\Pi_h(u_l, U_h(u_l))/\Pi_l(u_l, u_h)$ is increasing in $u_l$. This effect, which we call the “composition effect,” leads them to increase $u_h$ at faster rates at higher $u_l$.

To illustrate these two forces more clearly, consider the following optimality condition that any...
equilibrium menu \((u_l, U_h(u_l))\) must satisfy: \(^{33}\)

\[
U'_h(u_l) = \frac{\phi_l}{\phi_h} \frac{\Pi_h(u_l, U_h(u_l))}{\Pi_l(u_l)}
\] (26)

where \(\phi_h = (v_h - c_l)/(c_h - c_l)\) is the marginal cost of providing an additional unit of utility to type \(h\) sellers—i.e., \(\phi_h = \frac{d\Pi_h}{du_h}\)—and for notational convenience \(\Pi_l(u_l) \equiv \Pi_l(u_l, u_h)\). The first term, the incentive effect, is the ratio of the marginal costs of providing utility to the two types of sellers. Since this term is strictly less than 1, all else equal, the incentive effect leads to more aggressive competition for the low type and, therefore, to \(u_h\) rising more slowly than \(u_l\).

The second term, the ratio of profits, can be larger or smaller than 1, depending on \(u_l\). When \(u_l\) is close to the monopsony outcome, \(\Pi_h \approx 0\), so the composition effect is also less than 1 and we have \(U'_h(u_l) < 1\). However, as \(u_l\) approaches the upper bound \(v_l\), this second term overwhelms the incentive effect, resulting in \(U'_h(u_l) > 1\). In fact, one can show that \(\lim_{u_l \to v_l} \frac{\Pi_h}{\Pi_l} = \lim_{u_l \to v_l} U'_h(u_l) = \infty\). To see why, note that (applying l'Hôpital’s rule) reveals

\[
\lim_{u_l \to v_l} \frac{\Pi_h(u_l, U_h(u_l))}{\Pi_l(u_l)} = \lim_{u_l \to v_l} \frac{d\Pi_h(u_l, U_h(u_l))}{d u_l} = \lim_{u_l \to v_l} \phi_h U'_h(u_l) - \frac{v_h - c_h}{c_h - c_l}.
\] (27)

If this limit were finite, then

\[
\lim_{u_l \to v_l} \frac{\phi_l \Pi_h(u_l, U_h(u_l))}{\phi_h \Pi_l(u_l)} = \phi_l U'_h(v_l) - \frac{\phi_l v_h - c_h}{\phi_h c_h - c_l} < U'_h(v_l),
\] (28)

which implies that (26) cannot hold. In words, if the ratio of profits is finite in the limit, buyers have incentive to offer a lower \(u_h\). The only way to discourage such deviations is to make high types more profitable—in the limit, infinitely so. This is why \(x'_h(u_l) < 0\) close to the Bertrand outcome.

5.3 Reducing Information Asymmetries

We now study the welfare consequences of reducing informational asymmetries. This exercise sheds light on the implications of certain policy initiatives, as well as the effects of various technological innovations. For example, an important debate in insurance, credit, and financial markets centers around information that the informed party (the seller in our context) is required to disclose and the extent to

\(^{33}\)This equation combines the optimality condition (21) for \(u_l\), the corresponding optimality condition for \(u_h\),

\[
\frac{\pi f_h}{1 - \pi + \pi f_h} \Pi_h = \phi_h,
\]

and the strict rank-preserving property \(F_l(u_l) = F_h(U_h(u_l))\), which implies \(f_l = f_h U'_h(u_l)\).
Moreover, technological developments in these markets also have the potential to decrease informational asymmetries, as advanced record-keeping and more sophisticated scoring systems (e.g., credit scores) provide buyers with more and/or better information about sellers’ intrinsic types.

To study the effects of these changes, we introduce a noisy public signal \( s \in \{0, 1\} \) about the quality of each seller. The signal is informative, so that \( \Pr(s = 1|h) = \Pr(s = 0|l) > 0.5 \). Since the signal is publicly observed, the buyers may condition their offers on it, i.e., they offer separate menus for sellers with \( s = 0 \) and \( s = 1 \). Thus, the economy now has two subgroups, \( j \in \{0, 1\} \), with the fraction of high-quality sellers in subgroup \( j \) given by

\[
\mu_{hj} = \frac{\mu_h \Pr(s = j | h)}{\mu_h \Pr(s = j | h) + \mu_l (1 - \Pr(s = j | l))}.
\]

Note that the average across subgroups is equal to the unconditional fraction of high types, i.e., \( E[\mu_{hj}] = \mu_h \). The equilibrium outcome for each subgroup can be constructed using the procedure in Section 4 with the appropriate \( \mu_{hj} \). Welfare is then given by the average welfare across subgroups, i.e., \( E[W(\pi, \mu_{hj})] \).

When buyers do not observe a signal (or, equivalently, are not permitted to condition their offers on it), welfare is simply \( W(\pi, E[\mu_{hj}]) \). Hence, whether the signal increases or decreases welfare, respectively, depends on whether \( W(\pi, \mu_{h}) \) is convex or concave in \( \mu_h \) in the relevant region.

Before proceeding, two comments are in order. First, our focus is on the effect of a small increase in the information available to buyers; that is, we are interested in signals that induce a local mean-preserving spread around \( \mu_h \). Very informative signals always improve welfare—for example, if buyers receive a perfect signal about sellers’ types, then all gains from trade are realized —but this is not a very interesting or realistic experiment. Second, we focus on the region with \( \mu_h < \mu_0 \), so that \( \phi_l > 0 \), which is more tractable and shows interesting interactions between competition and additional information.

Moreover, in this region, \( W \) is linear in \( \mu_h \) when \( \pi = 0 \) or \( \pi = 1 \). Hence, imposing monopsony or perfect competition would lead us to the conclusion that additional information has no effect on welfare.

\[34\text{In insurance markets, these questions typically concern an individual’s health factors, both observable (e.g., age or gender) and unobservable (e.g., pre-existing conditions) to the insurance provider. In credit markets, similar questions arise with respect to observable characteristics that can legally be used in determining a borrower’s creditworthiness, as well as the amount of information about a borrower’s credit history that should be available to lenders (e.g., how long a delinquency stays on an individual’s credit history). In financial markets, the relevant issue is not only whether a seller discloses relevant information about an asset to a buyer, but also whether the payoff structure of the asset is sufficiently transparent for sellers to distinguish good from bad assets. For example, in order to support “sustainable securitisation markets,” the Basel Committee on Banking Supervision and the International Organization of Securities Commissions established a joint task force to identify criteria for “simple, transparent, and comparable” securitized assets. See http://www.bis.org/bcbs/publ/d304.pdf.}

\[35\text{See, e.g., Chatterjee et al. (2011) and Einav et al. (2013) for a description of how the emergence of standardized scoring systems in credit markets have radically changed lenders’ ability to assess a borrower’s creditworthiness.}

\[36\text{The restriction to a binary signal is only for simplicity. It is easy to introduce richer information structures.}

\[37\text{Numerical simulations suggest that additional information always reduces welfare when } \mu_h > \mu_0.\]
Proposition 6 shows that $W$ has a strictly convex region when $\pi$ is sufficiently low, implying that more information is beneficial when markets are close to (but not at) the monopsony benchmark. Alternatively, when markets are relatively (but not perfectly) competitive, $W$ has a strictly concave region, implying that more information actually reduces welfare.

**Proposition 6.** There exist $\pi, \pi \in (0, 1)$ such that: (i) for all $\pi \in (0, \pi)$, there exists $0 < \mu_h < \bar{\mu}_h < \mu_0$ such that $W$ is strictly convex on the interval $[\mu_h, \bar{\mu}_h]$; and (ii) for all $\pi \in (\pi, 1)$, there exists $0 < \mu_h' < \bar{\mu}_h' < \mu_0$ such that $W$ is strictly concave on the interval $[\mu_h', \bar{\mu}_h']$.

To see the intuition behind Proposition 6, recall from the previous subsection that trade with the high-quality seller (and thus welfare) is governed by the interaction of the incentive effect and the relative profit (or composition) effect. The consequences of more information can be understood in terms of these two forces, too, which depend on the severity of adverse selection. In particular, a lower $\phi_l$ drives down the first term in (26), which encourages more competition for low-quality sellers and, hence, boosts trade and welfare. Now, from (18), we see that $\phi_l$ is a concave function of $\mu_h$. Since the additional signal induces a mean-preserving spread of $\mu_h$, it results in a lower $\phi_l$ on average, which, *ceteris paribus*, increases trade. This mechanism makes more information desirable. The effect from relative profits goes in the opposite direction. In equilibrium, milder adverse selection raises profits from high types relative to low types, which increases $U'_h$ and hence decreases trade. Close to monopsony, since the incentive effect dominates, more information raises welfare. The opposite happens when $\pi$ is close to 1 and the effect on relative profits dominates.

### 5.4 Constrained Efficiency

The analysis above establishes that, for the case of $\phi_l > 0$, increasing competition and reducing information asymmetries can have non-monotonic effects on the equilibrium volume of trade, and hence on the (utilitarian) welfare measure. However, one might be concerned that this non-monotonicity is an artifact of the particular game we study, as opposed to a robust feature of markets with asymmetric information. To address this concern, we now derive a constrained efficient benchmark, taking as given both the information and search frictions, and show that the expected volume of trade in our equilibrium coincides with that of the constrained efficient allocation when adverse selection is severe.

Though much of the formal analysis is relegated to the Appendix, we sketch the key features of our constrained efficient benchmark here. The type of a seller $i \in [0, 1]$, which is private information, can be summarized by a tuple $\theta_i \in \Theta \equiv \{l, h\} \times \{0, 1\} \times \{0, 1\}$, where the first element indicates the quality
of the seller’s good, while the second and third elements equal 1 if the seller is matched with buyer 1 and buyer 2, respectively, and equal 0 otherwise. The type of buyer \( k \in \{1, 2\} \) can be summarized by a function \( m^k : [0, 1] \mapsto \{0, 1\} \), such that \( m^k(i) = 1 \) if buyer \( k \) is matched with seller \( i \) and 0 otherwise.

Given this specification, a direct mechanism prescribes a transfer of numeraire, \( t^k_i \in \mathbb{R} \), and a transfer of goods, \( x^k_i \in [0, 1] \), for each \( i \in [0, 1] \) and \( k \in \{1, 2\} \) based on the reported types.\(^{38}\) These transfers have to satisfy a number of constraints. The first is feasibility: given our assumption that trade can only occur between agents who are matched, it must be that \( x^k_i = t^k_i = 0 \) if \( m^k(i) = 0 \). The second is individual rationality: we assume that agents always have the option to play the game we analyze in Sections 2–4, should one of them reject the proposed mechanism, so that that their expected payoffs have to be at least as good as the payoffs they receive in the equilibrium. The third constraint is incentive compatibility: allocations have to induce sellers to truthfully reveal both the quality of their good and the buyers with whom they are matched. The final constraint is exclusivity: consistent with our benchmark model, we assume that \( x^1_i x^2_i = 0 \) for all \( i \in [0, 1] \), so that sellers may not transfer some of their good to both buyers.

A constrained efficient allocation maximizes a pareto-weighted sum of utilities (of buyers and each type of seller) subject to the constraints described above. We say that an equilibrium is constrained efficient if the associated expected utilities coincide with those induced by a constrained efficient allocation.

**Proposition 7.** If \( \phi_1 > 0 \) or \( \phi_1 < \phi_2 \), then the equilibrium is constrained efficient. If \( \phi_1 \in [\phi_2, 0] \), then the equilibrium is constrained inefficient.

Proposition 7 establishes that the equilibrium yields the same gains from trade and welfare as a constrained efficient mechanism when \( \phi_1 > 0 \), but not necessarily when \( \phi_1 \leq 0 \). In particular, when \( \phi_1 \in [\phi_2, 0] \) and expected trading volume of high-quality goods is less than 1, a benevolent planner can improve upon equilibrium allocations by inducing a greater degree of cross-subsidization from high- to low-quality sellers. The source of this inefficiency—that buyers’ incentives to cream skim high quality sellers limits equilibrium cross-subsidization—is similar to that which arises in many models with adverse selection and competition (see, e.g., Rothschild and Stiglitz, 1976; Guerrieri et al., 2010).

Most importantly, Proposition 7 reveals that a benevolent planner cannot propose a trading mechanism that would strictly increase trading volume or welfare, relative to our equilibrium allocation, given \( \phi_1 > 0 \) and \( \pi \in [0, 1] \). This is true despite the fact that the planner faces a weaker set of incentive constraints than the buyer: since seller \( i \) is matched with buyer \( k \) if and only if buyer \( k \) is matched with

\(^{38}\)The Revelation Principle applies immediately to this environment, so that restricting attention to direct mechanisms is without loss of generality.
seller i, the planner can easily design a mechanism that costlessly induces sellers to truthfully reveal the buyers with whom they are matched. Hence, the only relevant incentive constraints are those which induce sellers to truthfully reveal their quality.

Finally, Proposition 7 reports efficiency properties of equilibrium allocations for any degree of adverse selection and competition. However, while information frictions are often considered a primitive, one could argue that the degree of competition in a market is instead an outcome. A natural question, then, is whether the optimal level of competition arises in an environment where this level is determined by the choices of market participants. In the next section, we extend our analysis to study an environment where the market structure—summarized by $\pi$—is endogenous. We study the relationship between $\pi$ and the severity of adverse selection, and we ask whether a benevolent planner could increase welfare by influencing the degree of competition.\footnote{A related exercise is to consider interventions that mimic the effects of increasing or decreasing competition. In the working paper version, Lester et al. (2015b), we study what happens when the government enters a market suffering from adverse selection as a “large buyer,” as it has in, e.g., the markets for student loans, health insurance (where the government offers a “public option”), or certain financial assets (where the government considered using money from the Troubled Asset Relief Program to purchase asset-backed securities). We show that, by offering to buy any quantity at a fixed price, the government can increase sellers’ outside option and promote more competition, which recreates the effects of increasing $\pi$. Such an intervention can increase our measure of welfare only when both market power and the distortions arising from adverse selection are severe. Otherwise, in stark contrast to existing studies of such interventions in competitive environments (see, e.g., Tirole, 2012; Guerrieri and Shimer, 2014a), we show that such programs can be detrimental to welfare even if, in principle, the intervention makes non-negative profits.}

6  Endogenous Market Structure

In this section, we allow buyers a choice over how intensely they advertise their offers to sellers. This exercise has two benefits: the degree of competition will be endogenously determined; and the measure of sellers who are contacted by at least one buyer, or what is often called coverage, will also be endogenously determined. This allows us to study which features of the environment determine the market structure and the corresponding welfare implications.

Setting. Suppose that, in addition to choosing a menu of contracts to offer, each buyer $k \in \{1, 2\}$ must also choose the effort or intensity with which their offer will be advertised to sellers; exerting effort is costly, but increases the likelihood that each seller observes their offer. We can model this choice formally by assuming that buyer $k$ can choose the probability $\hat{\pi}^k$ that each seller observes his offer by incurring a cost $C(\hat{\pi}^k)$, which is a continuously differentiable, strictly increasing, and strictly convex function with $C(0) = C'(0) = 0$ and $C'(1) = \infty$.\footnote{Note that this implies a fraction $(1 - \hat{\pi}^1)(1 - \hat{\pi}^2)$ of sellers receive zero offers. This is the sense in which coverage is endogenous in the current setup, whereas we fixed this fraction (to zero) in our benchmark model.} Note that $\hat{\pi}^k$ represents a slightly different object than

\footnote{Note that this implies a fraction $(1 - \hat{\pi}^1)(1 - \hat{\pi}^2)$ of sellers receive zero offers. This is the sense in which coverage is endogenous in the current setup, whereas we fixed this fraction (to zero) in our benchmark model.}
\( \pi \) represented in our benchmark model, since it affects both competition and coverage. However, what is crucial is that—just like \( \pi \) in our earlier analysis—\( \hat{\pi}^{-k} \) is the conditional probability that a seller who buyer \( k \) meets has a second offer. Hence, in a symmetric Nash equilibrium, \( \hat{\pi}^k = \hat{\pi}^{-k} \equiv \hat{\pi} \) remains the key determinant of the level of competition.

**The buyer’s problem.** Taking as given the other buyer’s advertising intensity, \( \hat{\pi}^{-k} \), and the distribution of offers that he makes to sellers of type \( i \in \{l, h\} \), which we denote \( F_i^{-k}(u_i^{-k}) \), buyer \( k \) chooses a tuple \( (\hat{\pi}^k, u_l^k, u_h^k) \) to maximize

\[
\sum_{i \in \{l, h\}} \mu_i \left[ \hat{\pi}^k (1 - \hat{\pi}^{-k}) + \hat{\pi}^k \hat{\pi}^{-k} F_i^{-k} (u_i^k) \right] \Pi_i (u_l^k, u_h^k),
\]

subject to the same participation and incentive constraints described in the benchmark model, with \( \Pi_i (\cdot, \cdot) \) defined in (14)–(15).

Factoring out \( \hat{\pi}^k \) from (29), one can immediately see that the choice of \( \hat{\pi}^k \) and \( (u_l^k, u_h^k) \) are separable. Hence, given \( \hat{\pi}^{-k} \), the first order conditions on \( u_l^k \) and \( u_h^k \) are exactly as they were before (replacing \( \pi \) with \( \hat{\pi}^{-k} \)), while the first order condition determining the optimal choice of \( \hat{\pi}^k \) is

\[
C' (\hat{\pi}^k) = \sum_{i \in \{l, h\}} \mu_i \left[ 1 - \hat{\pi}^{-k} + \hat{\pi}^{-k} F_i^{-k} (u_i^k) \right] \Pi_i (u_l^k, u_h^k). \tag{30}
\]

In a symmetric equilibrium, where \( \hat{\pi}^1 = \hat{\pi}^2 \equiv \hat{\pi} \), equation (30) implies that the marginal cost of increasing \( \hat{\pi} \) is equal to the equilibrium profits characterized in Propositions 2 and 3. Since these profits are decreasing in \( \hat{\pi} \), the next result follows almost immediately.

**Proposition 8.** For any \( \phi_1 < 1 \), there exists a unique symmetric equilibrium, with \( \hat{\pi}^* \in (0, 1) \) and \( \{F_i^*(u_i)\}_{i \in \{l, h\}} \) as described in Propositions 2 and 3.

In Lemma 5, below, we offer comparative statics with respect to the fraction of high quality sellers, \( \mu_h \).\(^{41}\) Recall that there exists a \( \mu_0 \) such that \( \phi_1 \geq 0 \) if and only if \( \mu_h \leq \mu_0 \). We show that the equilibrium \( \hat{\pi}^* \) is U-shaped in \( \mu_h \), achieving a minimum at \( \mu_h = \mu_0 \).

**Lemma 5.** The equilibrium advertising intensity \( \hat{\pi}^* \) is decreasing in \( \mu_h \) when \( \mu_h < \mu_0 \) and increasing in \( \mu_h \) when \( \mu_h \geq \mu_0 \).

To understand the intuition, consider first the case of “severe adverse selection,” i.e., when \( \mu_h < \mu_0 \) or, equivalently, when \( \phi_1 > 0 \). In this region, once information rents are taken into account, the buyer’s

\(^{41}\)The same techniques we use to derive these results can be applied equally easily for other parameters, as well.
payoff from trading with low-quality sellers is larger than the payoff from trading with high-quality sellers (even if \( v_h - c_h > v_l - c_l \)). Thus, from the buyer’s perspective, an increase in \( \mu_h \) in this region actually worsens the pool of potential sellers and, as a result, buyers optimally choose a lower \( \hat{\tau} \). The opposite is true when \( \mu_h > \mu_0 \), where we say adverse selection is “mild.” In this region, after adjusting for information rents, it is relatively more profitable to trade with high quality sellers, and thus buyers optimally choose larger values of \( \hat{\tau} \) as the fraction of high quality sellers increases.

Lemma 5 has implications for the relationship between the composition of high- and low-quality sellers in a market and the (endogenous) level of competition that prevails. In particular, this result suggests that competition for customers should be strongest in markets with less uncertainty about sellers’ types (i.e., extreme values of \( \mu_h \)), and weakest in markets with more uncertainty about sellers’ types (i.e., intermediate values of \( \mu_h \)).

The model with endogenous \( \hat{\tau} \) also allows us to connect some of our welfare results to more concrete implications for policy. To see this, suppose \( C(\hat{\tau}^k) = Ac(\hat{\tau}^k) \) for some positive constant \( A > 0 \), and consider the effect of taxing buyers’ advertising intensities according to a proportional tax, \( \tau \hat{\tau} \). For simplicity, suppose all tax proceeds are then simply rebated to the agents. In Lemma 6, below, we establish that welfare is increasing in \( \tau \) in some regions of the parameter space. That is, a policy making it more costly for buyers to contact sellers can improve welfare.

**Lemma 6.** Suppose \( \phi_1 > 0 \). There exists an \( \tilde{A} > 0 \) such that welfare is increasing in \( \tau \) for all \( A < \tilde{A} \).

The result in Lemma 6 follows closely from the fact that welfare is hump-shaped in \( \hat{\tau} \), even taking into account that an increase in \( \hat{\tau} \) increases coverage. As a result, when \( A \) is sufficiently small, \( \hat{\tau}^* \) is large and a decrease in \( \hat{\tau}^* \)—brought about by an increase in \( \tau \)—causes welfare to rise.

**7 Large Markets and Meeting Technologies**

In this section, we show how our analysis and results extend to a more general environment in which there are an arbitrarily large number of buyers and sellers, and a more general meeting technology. In particular, suppose there is a measure \( b \) of buyers and a measure \( s \) of sellers. As in our benchmark model, buyers send out offers and sellers receive these offers. The meeting technology dictates the number of offers each buyer gets to send, and where these offers end up.

Formally, let \( \eta \) denote the (expected) number of offers that each buyer sends, and let

\[
\lambda = \frac{\eta b}{s}
\]
denote the ratio of offers to sellers. In addition, let \( P_n \) denote the probability that each seller receives \( n \in \{0\} \cup \mathbb{N} \) offers. A meeting technology, then, can be succinctly summarized by a pair \((\lambda, P_n)\). From a buyer’s perspective, a meeting technology implies that an offer he sends is received by a seller with \( n - 1 \) other offers with probability \( Q_n \), where
\[
np_n = \lambda Q_n \quad \text{for all } n \in \mathbb{N}.
\]

Following the convention in the literature, we let \( Q_0 = 1 - \sum_{n=1}^{\infty} Q_n \) denote the probability that an offer doesn’t reach a seller.

In what follows, we first show how to characterize the equilibrium for any meeting technology using the tools we developed in the two-buyer case. Then, we define the utilitarian welfare measure in this generalized environment, and study what happens when the meeting technology becomes “less frictional,” i.e., when sellers receive more offers (in a sense to be made precise). We show that there are two effects. First, as in our benchmark model, fewer frictions imply more competition, which can cause welfare to fall when the market is sufficiently close to perfect competition. Second, fewer frictions imply that more sellers receive at least one offer—i.e., that market coverage increases—which causes welfare to go up. Using several examples, we show that welfare continues to be maximized at an interior level of frictions, so long as the latter effect is not too strong.

### 7.1 Characterizing Equilibrium

As in our benchmark model, we restrict attention to symmetric equilibria, where \( \{F_i(u_i)\}_{i \in \{L,H\}} \) summarizes the distribution of menus being offered by buyers. Taking this distribution as given, an individual buyer makes an offer \((u_L, u_H)\) that solves
\[
\max_{u_L, u_H} \sum_{i \in \{L,H\}} \mu_i \left[ \sum_{n=1}^{\infty} Q_n F_i^{n-1}(u_i) \right] \Pi_i(u_L, u_H),
\]
where, again, \( \Pi_i(u_L, u_H) \) is defined in (14)–(15). Importantly, the objective in (32) can be re-written
\[
\left( \sum_{n=1}^{\infty} Q_n \right) \sum_{i \in \{L,H\}} \mu_i \left[ \frac{Q_1}{\sum_{n=1}^{\infty} Q_n} + \sum_{n''=2}^{\infty} \frac{Q_{n''}}{\sum_{n'=1}^{\infty} Q_{n'}} F_i^{n''-1}(u_i) \right] \Pi_i(u_L, u_H)
\]
or, equivalently,
\[
[1 - Q_0] \sum_{i \in \{L,H\}} \mu_i [1 - \pi + \pi G_i(u_i)] \Pi_i(u_L, u_H)
\]

---

42Note that this formulation of a meeting technology is slightly more general than what is commonly used in the existing literature (see, e.g., Eeckhout and Kircher, 2010), in the sense that we allow the “queue length” \( \lambda \) to depend on the meeting technology, whereas this is typically treated as a primitive.
where
\[ \tilde{\pi} = 1 - \frac{Q_1}{1 - Q_0} \] (34)
is the probability that an offer is received by a seller that has at least one other offer, conditional on being received by a seller, and
\[ G_i(u_i) = \frac{1}{\tilde{\pi}} \sum_{n=2}^{\infty} \frac{Q_n}{1 - Q_0} F_{i}^{n-1}(u_i) \] (35)
is the probability that the seller accepts the offer \( u_i \), given that they own an good of quality \( i \in \{l, h\} \).

Notice immediately that (33) has the same form as our objective function in the two-buyer case—replacing \( \pi \) with \( \tilde{\pi} \) and \( F_i(u_i) \) with \( G_i(u_i) \). As a result, our characterization of equilibrium in Propositions 2 and 3 is preserved and the distribution \( G_i(u_i) \) is uniquely defined in all regions of the parameter space. Moreover, from (35), it is easy to show that \( G_i(u_i) \) uniquely determines the distribution of offers made by buyers, \( F_i(u_i) \). Hence, with a large number of buyers and an arbitrary meeting technology, one can easily determine the type of contracts that are offered in equilibrium by comparing \( \phi_l \), which is unchanged in this general setting, to \( \phi_1 \) and \( \phi_2 \), which are updated by replacing \( \pi \) with \( \tilde{\pi} \); the distribution of offers that are made to each type of seller, \( F_i(u_i) \), which is the solution to (35); and the prices and quantities that are ultimately traded in equilibrium.

### 7.2 Competition, Coverage, and Equilibrium Gains from Trade

In our benchmark model, we studied the welfare effects of changing the probability that a seller received two offers, \( \pi \). In this section, we explore similar comparative statics within the context of a general meeting technology. In particular, we let \( P_n \) and \( \lambda \) (and hence \( Q_n \)) depend on a parameter \( \alpha \). This formulation is intentionally general: a change in \( \alpha \) could correspond to a change in the measure of buyers, a change in the expected number of offers per buyer, or a change in the technology that matches offers to sellers.

As in Section 6, we focus on the case where \( \phi_1 > 0 \) and define the utilitarian welfare measure
\[
W(\alpha) = \sum_{n=1}^{\infty} P_n(\alpha) \left[ \mu_h(v_h - c_h) \int_{x_h(u_l)} x_h(u_l) d(F_i^n(u_l)) + \mu_l(v_l - c_l) \right] + \sum_{i=l,h} \mu_i c_i
\]
As in our benchmark model, when \( \phi_1 > 0 \), the distribution \( G_i(u_i) \) solves the differential equation
\[
\frac{\tilde{\pi} g_l(u_l)}{1 - \tilde{\pi} + \tilde{\pi} G_l(u_l)} = \frac{\phi_1}{v_l - u_l}
\] (36)
with support \([c_l, \bar{u}_l(\alpha)]\) such that \( G_l(c_l) = 0 \) and \( G_l(\bar{u}_l(\alpha)) = 1 \).
Solving (36) and imposing equal profits implies that the mapping $U_h(u_1)$ must satisfy

$$
\left( \frac{v_1 - c_1}{v_1 - u_1} \right)^{\phi_1} \left[ \mu_l (v_1 - u_1) + \mu_h \Pi_h (u_1, U_h (u_1)) \right] = \mu_l (v_1 - c_1),
$$

exactly as in the case of two buyers. An immediate, and important, implication is that $x_h(u_1)$ is hump-shaped in $u_1$ and independent of $\alpha$. Hence, a change in $\alpha$ only affects the distribution of offers that are made, summarized by $F_l$, and the distribution of offers that sellers receive, summarized by $P_n$.

As a result, the effects of a change in $\alpha$ can be decomposed as follows:

$$
W' (\alpha) = \sum_{n=1}^{\infty} \frac{\partial P_n (\alpha)}{\partial \alpha} \left[ \mu_h (v_h - c_h) \int x_h(u_1) d (F^n_l (u_1; \alpha)) + \mu_l (v_l - c_l) \right] + \sum_{n=1}^{\infty} P_n (\alpha) \left[ \mu_h (v_h - c_h) \left( \frac{\partial \mu_l (v_l - c_l)}{\partial \alpha} \right) \int x_h(u_1) d (F^n_l (u_1; \alpha)) \right],
$$

where, for the purpose of clarity, we’ve made the dependence of $F_l$ on $\alpha$ explicit. The first term in the equation above was absent in our benchmark model, but captures a standard effect in models with frictions: the effect of a change in $\alpha$ on the set of sellers who are able to trade, or what we call the coverage effect. The second term captures the effect that we focused on in our benchmark model: the effect of a change in $\alpha$ on the distribution of offers, or what we call the competition effect.

For example, suppose increasing $\alpha$ leads to a first-order stochastic dominant (FOSD) shift in the number of offers that sellers receive. In this case, the coverage effect would be positive, since fewer sellers receive zero offers. However, the competition effect could be negative, since an increase in $\alpha$ leads to a FOSD shift in the distribution of offers $F_l$. As in our benchmark model, when $\alpha$ is sufficiently large, this shift puts more weight on the downward-sloping region of $x_h(u_1)$, thus reducing welfare.

Which of these two effects dominates typically depends on the details of the meeting technology. In what follows, we utilize several examples to further illustrate these two opposing forces, and to confirm that our results from the benchmark model—namely, that some frictions can increase welfare—remain true for certain popular meeting technologies even after accounting for the coverage effect.

**Examples of meeting technologies.** Consider first the Poisson meeting technology with $\lambda (\alpha) = \alpha$ and

$$
P_n (\alpha) = \frac{e^{-\alpha} \alpha^n}{n!}.
$$

This is perhaps the most popular meeting technology in the literature (see, e.g., Butters (1977) and Hall (1977) for early examples, and Burdett et al. (2001) and Shimer (2005) for more recent examples), and an increase in $\alpha$ clearly leads to a FOSD shift in the distribution of offers that sellers receive. We show
in the Appendix that when $\phi_l > 0$, there exists an $\alpha^*$ such that welfare is decreasing in $\alpha$ for all finite $\alpha > \alpha^*$. Therefore, as in our benchmark model, welfare is decreasing as the economy gets close to the frictionless benchmark, and hence welfare is maximized at an interior value of $\alpha$.

The same is not true, however, for all meeting technologies. For example, consider the Geometric meeting technology with $\lambda(\alpha) = \alpha/(1 - \alpha)$ and

$$ P_n(\alpha) = \alpha^n(1 - \alpha), $$

which was studied recently by, e.g., Lester et al. (2015a). Under this meeting technology, when $\phi_l > 0$, we show in the Appendix that the coverage effect always dominates the competition effect, so that welfare is increasing in $\alpha$. Intuitively, the coverage effect is relatively strong because the fraction of sellers who fail to receive an offer, $P_0$, falls slowly in $\alpha$ as $\alpha \to 1$, whereas the competition effect vanishes more quickly.

Note, however, that one can augment the Geometric meeting technology to ensure full coverage by setting $\lambda(\alpha) = \alpha/(1 - \alpha)$ and

$$ P_n(\alpha) = \alpha^{n-1}(1 - \alpha) $$

for $n \in \mathbb{N}$, with $P_0 = 0$. This specification removes the positive effects of increased coverage on welfare, leaving only the negative competition effect as $\alpha$ approaches 1. Hence, as in our benchmark model, in this case reducing frictions can again lead to welfare losses.

## 8 Additional Extensions and Robustness

In this section, we examine a few additional extensions of our framework, both to ensure the robustness of our results and to demonstrate that our framework is amenable to more applied work. First, we relax our assumption of linear utility to analyze the canonical model of insurance under private information. Second, we allow the degree of competition to differ across sellers of different quality. Last, we incorporate additional dimensions of heterogeneity, including horizontal and vertical differentiation.

### 8.1 A Model of Insurance

To start, we analyze a canonical model of insurance under private information, along the lines of Rothschild and Stiglitz (1976), and show that our main results—in particular, the structure of equilibrium menus and the non-monotonicity of welfare with respect to the degree of competition—extend beyond the linear, transferable utility environment.
A unit measure of agents with strictly increasing, strictly concave utility functions $w(c)$ face idiosyncratic income risk. Their income in normal times is $y$, but they also face the risk of an “accident,” which reduces their income by $d$. The accident is observable and contractible, but the probability of its occurrence, denoted $\theta_j$, $j \in \{b, g\}$, is private information. A fraction $\mu_b$ of agents are of type $b$ and face a higher risk of accident than type $g$ agents, i.e., $\theta_b > \theta_g$. Principals (the insurance providers) are risk-neutral, which implies that gains from trade are strictly positive for both types. The competitive structure is exactly the same as our baseline model: a fraction $1 - \pi$ of agents receive one offer and the remainder receive two.

A contract consists of a premium and a transfer to the agent in the event of an accident. Since trading is exclusive and the accident is observable, we can also think of the contract as directly offering a utility level in the normal and accident states. As before, we consider menus with two contracts, one for each type, i.e., $z = (u^n_b, u^n_a), (u^n_g, u^n_a)$ such that incentive and participation constraints are satisfied:

\[
\begin{align*}
(\text{IC}_j) & : \theta_j u^n_a + (1 - \theta_j) u^n_j \geq \theta_j u^a_j + (1 - \theta_j) u^n_j, \\
(\text{PC}_j) & : \theta_j u^n_a + (1 - \theta_j) u^n_j \geq \theta_j w(y - d) + (1 - \theta_j) w(y) \quad j \in \{b, g\}.
\end{align*}
\]

To solve for the equilibrium, we follow the same steps as in Section 4. The first step is to obtain the utility representation. It is straightforward to prove that, in all equilibrium menus, type $b$ agents are fully insured and $(\text{IC}_b)$ binds. This allows us to summarize equilibrium menus with a pair of expected utilities, $(u^b, u^g)$, and allocations given by the solution to the following system of equations:

\[
\begin{align*}
& u_b = u^n_b = u^m_b, \quad u_b = \theta_b u^a_g + (1 - \theta_b) u^n_g, \quad u_g = \theta_g u^a_g + (1 - \theta_g) u^n_g. \tag{37}
\end{align*}
\]

In a separating menu, the principal offers type $g$ agents less than full insurance: $u^a_g < u^n_g$ such that $(\text{IC}_b)$ binds. Define $C(u) \equiv w^{-1}(u)$ to be the principal’s cost of providing a utility level $u$. Note that $C'(u), C''(u) > 0$. Then, the objective of the principal is described by (12), where the type-specific profit functions satisfy

\[
\begin{align*}
\Pi_b (u_b, u_g) & = y - \theta_b d - C(u_b), \\
\Pi_g (u_b, u_g) & = y - \theta_g d - \theta_g C(u^a_g) - (1 - \theta_g) C(u^n_g).
\end{align*}
\]

Since $w$ is strictly increasing and concave, we can show that

\[
\frac{d\Pi_g (u_b, u_g)}{du_b} > 0, \quad \text{and} \quad \frac{d\Pi_g (u_b, u_g)}{du_g du_b} > 0.
\]

\footnote{Note that, in this application, the “buyers” of insurance are the ones with private information. To avoid confusion, we switch to a principal-agent description.}
The first inequality shows the effect of incentives: more surplus to type \textit{b} agents relaxes their incentive constraint, allowing the principal to earn higher profits from type \textit{g} agents. The second inequality shows that the marginal benefit of increasing the utility of type \textit{g} agents rises with the utility offered to type \textit{b} agents, implying the strict supermodularity of the profit function. In other words, the complementarity that was at the heart of the strict rank-preserving property in the linear model is present in this version as well. Using this property, we can extend the arguments in Proposition 1, implying that the marginal distributions \( F_j, j \in \{ b, g \} \) do not have any flat portions or mass points. Hence, Theorem 1 applies—equilibria are strictly rank-preserving—and can therefore be described by a distribution over utilities to type \textit{b} agents, \( F_b(u_b) \), and a strictly increasing function \( U_g(u_b) \). In Appendix A.6.1, we use the methods from Section 4 to derive the system of differential equations that characterize these functions.

Next, we consider the implications of competition for welfare. For brevity, we restrict attention to the region where all menus are separating and do not involve cross-subsidization. In this case, the consumption of type \textit{g} agents necessarily varies with the state; this imperfect insurance is the analogue of distortions in the quantity traded in the baseline model. The associated resource costs are thus a natural measure of the efficiency losses (relative to a full information benchmark) in this setting. For a menu offering \( u_b \) to type \textit{b} agents, this loss is given by

\[
L(u_b) = C(U_g(u_b)) - \left[ \theta_g C(U_g^n(u_b)) + (1 - \theta_g) C(U_g^a(u_b)) \right], \tag{38}
\]

where \( U_g, U_g^n, \) and \( U_g^a \) are equilibrium policy functions. Average losses in the economy are then

\[
\mathcal{L}(\pi) \equiv (1 - \pi) \int L(u_b) dF_b(u_b, \pi) + \pi \int L(u_b) dF_b(u_b, \pi)^2. \tag{39}
\]

In Appendix A.6.1, we show, using a numerical example, that \( L \) is \( U \)-shaped in \( u_b \), which then implies that \( \mathcal{L}(\pi) \) is minimized at an interior value of \( \pi \). Thus, in markets for insurance, increasing competition among providers can be detrimental for welfare.

8.2 Differential Competition Across Types

In our baseline model, we assume that the probability a seller receives one or two offers is the same for both types. In this subsection, we relax this assumption and allow \( \pi \) to vary across types, so that the probability a type \( j \) seller is captive is given by \( 1 - \pi_j \). We will show that both the structure of the equilibrium and its normative properties remain largely unchanged, with the caveat that, for some parameter values, the equilibrium distribution has mass points. For brevity, we restrict attention to the \( \phi_l > 0 \) case, where all equilibrium menus are separating and cross-subsidization does not occur.
When \( \pi_h > \pi_l \), the results in Proposition 1 go through unchanged, and thus the distribution functions \( F_l \) and \( F_h \) have continuous support and no mass points. This implies that the equilibrium satisfies the strict rank-preserving property and all menus attract the same fraction of noncaptive sellers. When \( \pi_l > \pi_h \), both distributions still have continuous supports, but \( F_l \) has a mass point if \( \pi_l \) is sufficiently large. The following proposition fully characterizes the unique equilibrium for both cases.

**Proposition 9.** If \( \frac{1 - \pi_l}{1 - \pi_h} < 1 - \phi_l \), then the unique equilibrium \( F_l \) has full mass at \( v_l \) and \( F_h \) is characterized by

\[
(1 - \pi_h + \pi_h F_h(u_h)) \Pi_h(v_l, u_h) = (1 - \pi_h) \Pi_h(v_l, c_h) .
\] (40)

If \( \frac{1 - \pi_l}{1 - \pi_h} \geq 1 - \phi_l \), then the unique equilibrium \( F_l \) satisfies

\[
\frac{\pi_l f_l(u_l)}{1 - \pi_l + \pi_l F_l(u_l)} \Pi_l(u_l) = 1 - \frac{1 - \pi_h + \pi_h F_l(u_l)}{1 - \pi_l + \pi_l F_l(u_l)} \left( \frac{\mu_h}{\mu_l} \right) \frac{v_h - c_h}{c_h - c_l}
\] (41)

and \( U_h \) is determined by the equal profit condition.

Equation (41) is similar in structure to (21). The key difference is that the right-hand side, which again measures the (net) marginal cost of providing a unit of surplus to the low type, has an additional term that adjusts for the differential probability that an offer is accepted by high types relative to low types. Naturally, this probability is small (i.e., the cost is large) when \( u_l \) is small and \( \pi_h \) is large.

The construction of equilibrium follows the strategy in Section 4. The ordinary differential equation in (41), with the boundary condition \( F_l(c_l) = 0 \), can be solved for \( F_l \). Given \( F_l \), the equal profit condition pins down \( U_h \). The properties of the equilibrium—both positive and normative—are also similar to the baseline model. In particular, \( x_h \) is non-monotonic in \( u_l \) which, as before, has interesting implications for the relationship between welfare and competition.

Figure 5 illustrates the effects of varying competition for each type separately. The left panel varies \( \pi_h \), holding \( \pi_l \) fixed, and shows that more competition for high-quality sellers always reduces welfare; intuitively, more surplus to high-quality sellers tightens the incentive constraints and reduces trade. The right panel varies \( \pi_l \), holding \( \pi_h \) fixed, which has two effects (exactly as in section 5.2). First, it increases surplus to low-quality sellers, which relaxes incentive constraints and increases trade with high-quality sellers. Second, it makes low-quality sellers relatively less attractive to buyers, inducing them to compete more aggressively for the high-quality seller, reducing trade. These two competing forces lead to a non-monotonic relationship between \( \pi_l \) and welfare, provided \( \pi_l \) is sufficiently high.\(^{44}\)

\(^{44}\)When \( \pi_h \) is low, we enter the region with mass points before the second (negative) effect begins to dominate. Since a mass point equilibrium puts full mass at \( v_l \), increasing \( \pi_l \) beyond this point has no effect on welfare.
8.3 Differentiation and Multidimensional Heterogeneity

In this section, in order to enhance the applicability of our framework to applied work, we introduce various types of additional heterogeneity: across buyers, across contracts, and across sellers. In various ways, these generalizations break the stark relationship between a seller’s type, the offer she accepts, and the rank of that offer within the distribution of all offers. The cost of these generalizations is some degree of tractability, though we argue that, in most cases, the properties and characterization of equilibria are very similar to the baseline framework. For brevity, we restrict attention to the region of the parameter space where almost all equilibrium menus are separating and not cross-subsidizing.

Horizontal differentiation across buyers. Consider first the possibility that buyers are horizontally differentiated. Specifically, as in the discrete choice model of McFadden (1974), we assume that the payoff to a seller of type $i$ from a contract $(x, t)$ offered by buyer $k$ is

$$u_{ik} = (1 - x) c_i + t + \epsilon_k = u_i + \epsilon_k,$$

where $\epsilon_k$ is a buyer-specific preference shock drawn from a continuous distribution $H$ with support $[\epsilon, \bar{\epsilon}]$. Note that $\epsilon$ is the same for both seller types, so it has no effect on the incentive constraints. Hence, we may once again represent each equilibrium menu by a utility pair $(u_l, u_h)$. A captive seller accepts this menu if $u_{ik}$ is greater than her outside option, $c_i$, which occurs with probability

$$\hat{F}_i^c (u_i) = \int_{c_i - u_i}^{\epsilon} dH(e) = 1 - H(c_i - u_i).$$

(42)
A noncaptive seller of type $i$ accepts this menu if $u_i + \epsilon > \max(u'_i + \epsilon', c_i)$, which occurs with probability

$$
\tilde{F}_i^{nc}(u_i) = \int_{u_i}^{\epsilon} \int_{c_i - u_i}^{\epsilon - u'_i} \left( \int_{\epsilon}^{u'_i + \epsilon - u'_i} dH(\epsilon') \right) dH(\epsilon) dF_i(u'_i)
$$

(43)

where $F_i$ is the marginal distribution of utilities offered to type $i$ sellers in equilibrium. Setting $M_i(u_i) = (1 - \pi) \tilde{F}_i^{c}(u'_i) + \pi \tilde{F}_i^{nc}(u'_i)$, we can write the buyer’s problem as

$$
\max_{u_{il}, u_{ih}} \sum_{i \in \{l, h\}} M_i(u'_i) \Pi_i(u'_l, u'_h).
$$

(44)

In a separating equilibrium, optimality with respect to $u_l$ requires

$$
\frac{m_l(u_l)}{M_l(u_l)} (v_l - u_l) = \phi_l.
$$

(45)

In other words, the link between the trading probability and the utility offered to the low-quality seller is exactly the same as in our baseline framework, and all of our results go through with respect to the key equilibrium objects $M_l$ and $M_h$. The only caveat is that recovering the underlying distribution of offers $F_l$ and $F_h$, which are informative about prices and allocations, typically requires numerical methods.\(^{45}\)

**Horizontal differentiation across contracts.** The extension above allows for the possibility that a seller accepts a contract from the “wrong” buyer, i.e., accepts $u_i$ even though a contract $u'_i > u_i$ was available. In this section, we allow for the possibility that a seller accepts the “wrong” contract within a menu, i.e., accepts $u_{-i}$ even though her type is $i$. In particular, suppose that a fraction $\delta$ of low-quality sellers accept the contract intended for a high-quality seller. It is possible to microfound this as a form of “tremble,” or as arising from other unmodeled contract features that cause some low-quality sellers to prefer the contract with lower quantity and higher price.\(^{46}\)

For example, the high price contract might carry other benefits, such as better customer service, that are valued by some low-quality sellers (but not others).

Let $\tilde{v}_h = \frac{\mu_h v_h + \mu_i \delta v_i}{\mu_h + \mu_i \delta}$ be the average value (to the buyer) of goods held by agents who take the contract intended for the high type. We assume that $\delta$ is sufficiently small so that $\tilde{v}_h > c_h$. The expected profits of the buyer, conditional on trade, are then given by $\bar{\Pi}_i(u_l, u_h) = \tilde{v}_h - \left( \frac{\tilde{v}_h - c_l}{c_h - c_l} \right) u_h + \left( \frac{\tilde{v}_h - c_h}{c_h - c_l} \right) u_l$. As in our baseline model, the FOC for $u_l$ and the equal profit condition pin down $F_l$ and $U_h$:

$$
\frac{\pi F_l(u_l)}{1 - \pi + \pi F_l(u_l)} (v_l - u_l) = 1 - \frac{\mu_h + \mu_i \delta}{\mu_l (1 - \delta)} \left( \frac{\tilde{v}_h - c_l}{c_h - c_l} \right) = \phi_l,
$$

(46)

$$
(1 - \pi) \mu_l \delta (v_l - c_l) = (1 - \pi + \pi F_l(u_l)) \left[ \mu_l (1 - \delta) (v_l - u_l) + (\mu_h + \mu_l \delta) \bar{\Pi}_i(u_l, u_h) \right].
$$

(47)

\(^{45}\)The differential equation in (45), along with the equal profit condition and the system of integral equations in (42) -- (43) must be solved jointly for $F_l$, and this system is only analytically tractable under special assumptions on the distribution $H$.

\(^{46}\)For simplicity, we make two additional assumptions. First, a captive low-quality seller still chooses the more attractive menu, even when she takes the contract intended for the high-quality seller. Second, we assume that the buyer does not (or cannot) try to use contract terms to separate out these low-quality sellers.
Note that these equations are very similar to (21)–(22), with \( \tilde{\Pi}_h \) and \( \tilde{\phi}_l \) replacing \( \Pi_h \) and \( \phi_l \). Accordingly, the characterization and other results in the preceding sections directly extend.

**Vertical differentiation across buyers.** Suppose now that sellers attach a higher value to trading with certain buyers, i.e., that the utility of a type \( i \) seller from accepting a contract \((x, t)\) from buyer \( k \in \{1, 2\} \) is given by \( c_i (1 - x) + t + B^k \), where \( B^1 \equiv B > 0 \) and \( B^2 \) is normalized to zero.\(^{47}\) This implies that the cost of delivering utility to sellers is lower for buyer 1 or, equivalently, his profits are higher than those of buyer 2, i.e., \( \Pi^1_i (u_l, u_h) = \Pi^2_i (u_l, u_h) + B \). Not surprisingly, in this environment, the equilibrium distribution of menus is also asymmetric. Let \( F^k_i (u_i) \), \( k \in \{1, 2\} \) denote the marginal distribution of utilities offered by buyer \( k \) to type \( j \) sellers. In Appendix A.6.4, we characterize an equilibrium in which these distributions satisfy the strict rank-preserving property, except at the lower bound of the support, where \( F^2_i \) has a mass point.\(^{48}\)

**Multidimensional seller heterogeneity.** Finally, our baseline framework posits a tight connection between the valuations of the seller and the buyer. While this is a natural assumption when sellers are heterogeneous along a single dimension—asset quality—it is also natural to consider the case in which sellers have heterogeneous preferences as well.\(^{49}\) A simple way to incorporate this additional heterogeneity into our analysis is to assume that a seller’s type is a tuple \((c, \tilde{v})\), with \( c \in \{c_h, c_l\} \) denoting the seller’s valuation for her asset and \( \tilde{v} \in \{\tilde{v}_h, \tilde{v}_l\} \) denoting the buyer’s valuation. This allows for the possibility that some high (low) quality assets are held by sellers who, for idiosyncratic reasons, have a low (high) valuation for them. In an asset market interpretation, for example, this could arise from heterogeneity in discount rates or liquidity needs. Let \( \mu_{ij} \) denote the proportion of sellers of type \((c, \tilde{v})\).

We can show that it is not possible for buyers to separate sellers with the same \( c \) but different \( \tilde{v} \)’s. Let \( \mu_i = \sum_j \mu_{ij} \) denote the fraction of sellers with valuation \( c_i \), \( i \in \{h, l\} \) and \( v_i = \frac{\sum_j \mu_{ij} \tilde{v}_j}{\mu_i} \) denote the average value (to the buyer) of the assets held by sellers of type \( i \). Assuming that gains from trade are positive, so that \( c_i < v_i \), it is easy to see that our analysis of the baseline model goes through exactly. In other words, additional preference heterogeneity changes the interpretation of buyer values in our baseline model, but otherwise leaves the analysis unchanged.

\(^{47}\)Equivalently, and more consistent with our earlier interpretation, one could imagine a measure of buyers, with a fraction of each type \( k \in \{1, 2\} \). The simplification here implies that a noncaptive seller will always have one offer from a type 1 buyer and one from a type 2 buyer, though this could be relaxed.

\(^{48}\)Our analysis requires one additional assumption: a seller who is indifferent between two menus chooses the one offered by buyer 1. The resulting system of differential equations can be solved numerically to obtain the equilibrium distributions.

\(^{49}\)See, for example, Finkelstein and McGarry (2006), Chang (2012), and Guerrieri and Shimer (2014b).
9 Conclusion

In their survey of the literature on insurance markets, Einav et al. (2010a) note that, despite substantial progress in understanding the effects of adverse selection,

“there has been much less progress on empirical models of insurance market competition, or on empirical models of insurance contracting that incorporate realistic market frictions. One challenge is to develop an appropriate conceptual framework. Even in stylized models of insurance markets with asymmetric information, characterizing competitive equilibrium can be challenging, and the challenge is compounded if one wants to allow for realistic consumer heterogeneity and market imperfections.”

In this paper, we overcome this challenge and develop a tractable, unified framework to study adverse selection, screening, and imperfect competition. We provide a full analytical characterization of the unique equilibrium, and use it to study both positive and normative issues.

Going forward, our framework can be exploited and extended to address a variety of important issues, both applied and theoretical. On the applied side, our equilibrium provides a new structural framework that can be used to jointly identify the extent of adverse selection and imperfect competition in various markets, and to study how the interaction of these two frictions affects the distribution of contracts, prices, and quantities that are traded. On the theoretical side, there are several obvious extensions to pursue. For example, one natural extension is to study the analog of our model with nonexclusive contracts; though this would complicate the analysis considerably, it would also make our framework suitable to analyze certain markets where exclusivity is hard to enforce. Finally, while we focus on screening as a mechanism for coping with adverse selection, of course there are other mechanisms as well, such signalling and reputation. It would be interesting to study how these mechanisms interact with competition. We leave these exercises for future work.

\[50\text{See, e.g., Bagwell and Ramey (1993).}\]
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Appendices

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A Omitted Proofs

This section contains proofs of the results presented in the main text.

A.1 Proofs from Section 3

A.1.1 Proof of Lemma 1

**Proof.** Both results are similar to existing results (see, for example, Dasgupta and Maskin (1986)), and thus we keep the exposition brief. To establish that \( x_1 = 1 \) in all equilibrium menus, suppose by way of contradiction that some equilibrium menu \( z = (z_l, z_h) \) has \( x_1 < 1 \) and \( t_1 \in \mathbb{R}_+ \), yielding a low-quality seller utility \( u_1 \). Now, consider a deviation \( z' = (z'_l, z'_h) \) with \( x'_l = x_1 + \epsilon \) for \( \epsilon \in (0, 1 - x_1) \) and \( t'_1 = t_1 + \epsilon c_1 \). Note that \( u'_1 = u_1 \), so that \( z_1 \) and \( z'_1 \) are accepted with the same probability, but

\[
x_1 v_l - t_1 < x_1 v_l - t_1 + \epsilon (v_l - c_1) = x'_l v_l - t'_1,
\]

so that \( z'_1 \) earns the buyer a higher payoff when it is accepted, implying existence of a profitable deviation. Therefore, no equilibrium menu features \( x_1 < 1 \).

To establish that a low-quality seller’s incentive compatibility constraint binds in all equilibrium menus, suppose by way of contradiction that some equilibrium menu \( z = (z_l, z_h) \) has \( t_1 > t_h + c_l (1 - x_h) \). Now, consider a deviation \( z' = (z'_l, z'_h) \) with \( x'_h = x_h + \epsilon \) and \( t'_h = t_h + \epsilon c_h \) for \( \epsilon \in \left(0, \frac{t_1 - t_h - c_l (1 - x_h)}{c_h - c_l}\right) \), which is a nonempty interval by assumption. The upper bound on \( \epsilon \) ensures that the incentive compatibility constraint on type \( l \) sellers is not violated. In addition, note that \( u'_h = u_h \), so that \( z_h \) and \( z'_h \) are accepted with the same probability, but

\[
x_h v_h - t_h < x_h v_h - t_h + \epsilon (v_h - c_h) = x'_h v_h - t'_h,
\]

so that \( z'_h \) earns the buyer a higher payoff when it is accepted, implying existence of a profitable deviation. Therefore, in all equilibrium menus, the type \( l \) seller’s incentive constraint binds. 

A.1.2 Proof of Proposition 1 and Lemma 2

We prove the proposition through the following sequence of lemmas.

**Lemma 7.** \( F_h(\cdot) \) has no flats.

**Proof.** Suppose by way of contradiction that \( F_h(\cdot) \) is flat in an interval \((u_{h1}, u_{h2})\). In other words, there exists \((u_{12}, u_{h2}) \in \text{Supp}(F_l) \times \text{Supp}(F_h)\) such that, for some \( \epsilon > 0 \), the distribution \( F_h \) satisfies \( F_h(u_{h2}) = F_h(u_{h2} - \epsilon) \) for all \( \epsilon \in [0, \delta] \). We prove that there must exist a profitable deviation. The particular deviation we construct depends on whether \( u_{12} < u_{h2} \) or \( u_{12} = u_{h2} \) and whether \( F_l \) is flat on an interval containing \( u_{12} \) or not. We consider each relevant case in turn:

1. Suppose that \( u_{12} < u_{h2} \). In this case, a deviation to \((u_{12}, u_{h2} - \epsilon')\) with \( \epsilon' < \delta \) is feasible and must be profitable because such a deviation increases profits earned from trading with \( h \) types but does not change the fraction of \( h \) types attracted.

2. Suppose that \( u_{12} = u_{h2} \) and \( F_l \) is flat below \( u_{12} \). In this case, a deviation of the form \((u_{12} - \epsilon', u_{h2} - \epsilon')\) for a small but positive \( \epsilon' \) is profitable since it increases profits per trade (from both \( l \) and \( h \) type sellers) but does not change the fraction of either type attracted.

3. Suppose \( u_{12} = u_{h2} \) and \( F_l \) is not flat below \( u_{12} \). Such a situation is depicted in Figure 6. Point \( A \) represents the contract \((u_{12}, u_{h2})\). Since \( F_h \) is flat by assumption, the area between the two red dashed lines must not contain any equilibrium menu. Since \( F_l \) is not flat below \( u_{12} \) by assumption
and there are no menus in the area between the red dashed lines, an equilibrium contract must exist in the region where the point \( D \) is located; recall, since \( u_h \geq u_l \), the point \( D \) cannot lie below the lower red dashed line. Let point \( D \) represent such an equilibrium menu. In addition, let \( B \) represent a menu with the same offer to the low type as \( D \) but offers \( u_h^2 \) to the high type. Similarly, let \( C \) represent a menu with the same offer to the low type as \( A \) and the same offer to the high type as \( D \).

For any distributions, \( F_l \) and \( F_h \), the profit function, \( \Pi(u_l, u_h) \) is weakly supermodular so that

\[
\Pi_A + \Pi_D \leq \Pi_C + \Pi_B.
\]

Since both \( D \) and \( A \) are offered in equilibrium, we must have that \( \Pi_A = \Pi_D \geq \Pi_C, \Pi_B \). This implies that \( \Pi_A = \Pi_B \). Additionally, since \( F_h \) is flat between \( B \) and \( E \) (and these menus offer the same \( u_l \)), it must be that \( \Pi_E > \Pi_B \). Therefore, this is a profitable deviation.

![Figure 6: A graphical illustration of why \( F_l \) cannot be flat.](image)

**Lemma 8.** \( F_l (\cdot) \) has no flats.

**Proof.** Suppose by way of contradiction that \( F_l \) is flat in an interval \((u_{l1}, u_{l2})\). Without loss of generality, we can complete the measure \( \Phi \) to include menus with first element given by \( u_{l1} \) and \( u_{l2} \). Since the profit function is weakly supermodular, then the policy correspondence must be weakly increasing. Now consider the policy correspondences \( U_h (u_{l1}) \) and \( U_h (u_{l2}) \). Note that \( Cl(U_h (u_{l1})) \) and \( Cl(U_h (u_{l2})) \) cannot be disjoint—if they were, then there would be a flat in the support of \( F_h \), which contradicts Lemma 7. Let \( \hat{u}_h \) be a common value in these two sets. We present a depiction of such a situation in Figure 7 below.

Holding \( \hat{u}_h \) fixed, the profit function must be linear over the set \((u_{l1}, u_{l2})\) since \( F_l (\cdot) \) is flat by assumption. Therefore, all the menus on the line \( AB \) must also deliver profits equal to equilibrium profits. However, since profits earned from trading with \( h \) types are increasing in \( u_l \), the marginal benefit of a change in \( u_h \) is changing along the line \( AB \). As a result, it is possible to construct an upward or downward deviation along \( AB \) that increases profits, implying existence of a profitable deviation.

**Lemma 9.** \( \Phi \) has no mass point.
Figure 7: A graphical illustration of why $F_l$ cannot be flat.

**Proof.** Suppose by way of contradiction that $\Phi$ has a mass point at the menu $(u_l, u_h)$. Let $m$ denote the mass at this menu. Since for any such menu, a deviation of the form $(u_l + \varepsilon_1, u_h + \varepsilon_2)$ for small $\varepsilon_1, \varepsilon_2$ (one of which is positive or negative) must be feasible, profits earned from the mass of sellers attracted to such deviation must be zero:

$$\mu_l \pi \frac{m}{2} \Pi_l (u_l) + \mu_h \pi \frac{m}{2} \Pi_h (u_l, u_h) = 0.$$ 

If the menu $(u_l, u_h)$ is interior to the constraint set—that is, if $c_h - c_l > u_h - u_l > 0$—then a simple deviation along $u_l$ or $u_h$ will be feasible and profitable. However, it is possible that $(u_l, u_h)$ is on the boundary of the set and, as a result, not all deviations are feasible. There are two relevant possibilities:

1. Suppose that the menu with mass, $(u_l, u_h)$, satisfies $u_h = u_l + c_h - c_l$. In such a case, the menu features no trade with the high type. Therefore, it must be that $\Pi_h \leq 0$. Since equilibrium profits are strictly positive, it must be that $\Pi_l > 0$. Hence, an infinitesimal increase in $u_l$, which is feasible, increases profits.

2. Suppose that the menu with mass, $(u_l, u_h)$ satisfies $u_h = u_l$. Then $(u_l, u_h)$ is a pooling menu. Therefore, the profits from the high type must be positive. As a result, the buyer offering this contract could increase profits with an infinitesimal increase in $u_h$ (which would attract a mass of high types), while holding $u_l$ constant.

**Lemma 10.** $F_h (\cdot)$ does not have a mass point.

**Proof.** Suppose by way of contradiction that $F_h$ has a mass point. From Lemma 9, we know that this mass point could not have been created from a mass point in $\Phi$. Therefore, if $F_h$ has a mass point at $\hat{u}_h$, it must be that a positive measure set of the form $\{ (u_l, \hat{u}_h) \}$ exists. Figure 8, depicts this possibility.

Note that at one of the points on the line, profits from the $h$ type, $\Pi_h (u_l, \hat{u}_h)$ must be non-zero since $\Pi_h$ is strictly increasing in $u_l$. Therefore, a small deviation upward or downward increases profits; this implies existence of a profitable deviation and yields the necessary contradiction.

To show that $F_l$ has no mass points, we make use of the strict supermodularity of the profit function, which only relies on the continuity of $F_h$. We therefore provide a proof of the strict supermodularity of the profit function here.
Proof of Lemma 2. Suppose \( u_{12} > u_{11} \) and \( u_{h2} > u_{h1} \). Then, letting \( \xi_1 \equiv \frac{v_h - c_h}{c_h - c_l} > 0 \) and \( \xi_2 \equiv \frac{v_h - c_l}{c_h - c_l} > 0 \),

\[
\begin{align*}
\Pi(u_{11}, u_{h2}) - \Pi(u_{11}, u_{h1}) &= \mu_h [(1 - \pi + \pi F_h(u_{h2}))\Pi_h(u_{11}, u_{h2}) - (1 - \pi + \pi F_h(u_{h1}))\Pi_h(u_{11}, u_{h1})] \\
&= \mu_h [(1 - \pi + \pi F_h(u_{h2})) [v_h + \xi_1 u_{11} - \xi_2 u_{h2}] - (1 - \pi + \pi F_h(u_{h1})) [v_h + \xi_1 u_{11} - \xi_2 u_{h1}]] \\
&< \mu_h [(1 - \pi + \pi F_h(u_{h2})) [v_h + \xi_1 u_{12} - \xi_2 u_{h2}] - (1 - \pi + \pi F_h(u_{h1})) [v_h + \xi_1 u_{12} - \xi_2 u_{h1}]] \\
&= \Pi(u_{12}, u_{h2}) - \Pi(u_{12}, u_{h1}),
\end{align*}
\]

where the inequality follows from the fact that \( F_h \) is strictly increasing, and hence

\[
\pi \xi_1(u_{12} - u_{11})[F_h(u_{h2}) - F_h(u_{h1})] > 0. \]

Lemma 11. \( F_l \) is continuous except possibly at \( v_l \).

Proof. Suppose by way of contradiction that \( F_l \) is not continuous and thus has a mass point at some \( \hat{u}_l \). Again, by Lemma 9, it must be that a positive measure set of the form \( S = \{(\hat{u}_l, u_h)\} \) exists. It is immediate that \( \Pi_l(\hat{u}_l) = 0 \); otherwise, it would be profitable to increase or decrease \( u_l \) by \( \varepsilon \) if \( \Pi_l(\hat{u}_l) > 0 \) or \( \Pi_l(\hat{u}_l) < 0 \), respectively. If \( \Pi_l(\hat{u}_l) = 0 \), then it must be \( \hat{u}_l = v_l \).}

A.2 Proofs from Section 4

A.2.1 Proof of Proposition 2

Proof. We first show that the equilibrium allocations constructed in (21) and (22) are indeed separating and interior. Our construction ensures that local deviations are not profitable. Below we prove that the global deviations are not profitable as well.

Verifying Allocations are Separating and Interior. Note that the solution to the differential equation in (22), together with boundary condition \( F_1(c_1) = 0 \), must satisfy

\[
1 - \pi + \pi F_1(u_1) = (1 - \pi) (v_l - c_1)^{\Phi_l} (v_l - u_1)^{-\Phi_l}. \tag{48}
\]
Therefore, from (22), \( U_h(u_1) \) must satisfy
\[
U_h(u_1) = \frac{1}{\mu_h \frac{v_h - c_1}{c_h - c_1}} \left[ \mu_h v_h + \mu_1 v_1 - \mu_1 \phi_1 u_1 - \mu_1 (v_1 - c_1)^{1 - \phi_1} (v_1 - u_1)^{\phi_1} \right].
\]

For the allocation to be separating, we must verify that
\[
 u_1 + c_h - c_1 \geq U_h(u_1) > u_1, \forall u_1 \in \text{Supp}(F_1)
\]
(49)
where
\[
\text{Supp}(F_1) = \left[ c_1, v_1 - (1 - \pi) \frac{1}{\phi_1} (v_1 - c_1) \right].
\]
The second inequality in (49), \( U_h(u_1) > u_1 \), is satisfied if and only if
\[
\mu_h v_h + \mu_1 v_1 > \mu_1 (v_1 - c_1)^{1 - \phi_1} (v_1 - u_1)^{\phi_1} + u_1
\]
(50)
for all \( u_1 \in \text{Supp}(F_1) \). Let \( H(u_1) \) denote the right-hand side of (50). We argue that \( H(\cdot) \) is strictly concave and attains its maximum at a value \( u_1^* \in [c_1, v_1] \) with \( H(u_1^*) < \mu_h v_h + \mu_1 v_1 \), implying that (50) is satisfied for all \( u_1 \in \text{Supp}(F_1) \). To see this, note that
\[
H'(u_1) = -\phi_1 \mu_1 (v_1 - c_1)^{1 - \phi_1} (v_1 - u_1)^{{\phi_1} - 1} + 1
\]
(51)
\[
H''(u_1) = \phi_1 (1 - \phi_1) \mu_1 (v_1 - c_1)^{1 - \phi_1} (v_1 - u_1)^{\phi_1 - 2} < 0,
\]
(52)
where the inequality in (52) is implied by the fact that \( 0 < \phi_1 < 1 \). Also, since \( \phi_1 < 1 \), \( H'(v_1) = -\infty \) and \( H'(c_1) = 1 - \phi_1 \mu_1 > 0 \), so that the maximum of \( H(u_1) \) is attained on the interior of \([c_1, v_1]\).

The function \( H(u_1) \) is maximized at \( u_1^* \) given by
\[
 u_1^* = v_1 - (\phi_1 \mu_1) \frac{1}{1 - \phi_1} (v_1 - c_1)
\]
with
\[
H(u_1^*) = v_1 + (v_1 - c_1) \mu_1 \frac{1}{1 - \phi_1} \frac{\phi_1}{\phi_1} [1 - \phi_1].
\]
Since \( c_h \geq v_1 \) and \( \phi_1 < 1 \), it is immediate that
\[
(\phi_1 \mu_1) \frac{\phi_1}{1 - \phi_1} < 1 \leq \frac{(c_h - c_1) (v_h - v_1)}{(v_1 - c_1) (v_h - c_h)},
\]
which implies
\[
(v_1 - c_1) \mu_1 (\phi_1 \mu_1) \frac{\phi_1}{1 - \phi_1} \frac{v_h - c_h}{c_h - c_1} < \mu_h (v_h - v_1).
\]
Hence,
\[
(v_1 - c_1) \mu_1 (\phi_1 \mu_1) \frac{\phi_1}{1 - \phi_1} (1 - \phi_1) < \mu_h (v_h - v_1)
\]
and
\[
\max_{u_1 \in [c_1, v_1]} H(u_1) = H(u_1^*) = v_1 + (v_1 - c_1) \mu_1 (\phi_1 \mu_1) \frac{\phi_1}{1 - \phi_1} (1 - \phi_1) < \mu_h (v_h - v_1) + v_1
\]
as needed.

We now establish that the first inequality in (49) is true, which requires showing that
\[
\frac{\mu_h v_h + \mu_1 v_1 - \mu_1 \phi_1 u_1 - \mu_1 (v_1 - c_1)^{1 - \phi_1} (v_1 - u_1)^{\phi_1}}{\mu_h \frac{v_h - c_1}{c_h - c_1}} \leq u_1 + c_h - c_1,
\]
or, equivalently,

\[ \mu_h c_l + \mu_l v_l \leq u_l + \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l}, \forall u_l \in \text{Supp} (F_l) \subset [c_l, v_l]. \]  

(53)

Since, the right side of (53) is a concave function, it takes its minimum values at the extremes of the interval \([v_l, c_l]\). These values are given by \(v_l\) and \(\mu_l v_l + \mu_h c_l\), both of which are at least as large as the left side of (53). Hence, (53) must be satisfied for all \(u_l \in [v_l, c_l]\), as needed.

**Global Deviations.** Note that our conditions (21) and (22) imply that local deviations with respect to \(u_h\) and \(u_l\) are not profitable. It, thus, remains to show that, for all \((u'_l, u'_h)\), \(\Pi (u'_l, u'_h) \leq \mu_l (1 - \pi) (v_l - c_l)\).

We consider two types of deviations:

1. Consider first deviation menus with \(u'_h > \max \text{Supp} (F_h) = \bar{u}_h\). Such deviations attract all type h sellers, so that \(1 - \pi + \pi F_h (u'_h) = 1\). If \(u'_l > \max \text{Supp} (F_l) = \bar{u}_l\), then the profits from this menu are given by

\[ \mu_l (v_l - u'_l) + \mu_h \Pi_h (u'_l, u'_h). \]

Since \(\phi_l > 0\), the above function is decreasing in \(u'_l\) and \(u'_h\), and therefore

\[ \mu_l (v_l - u'_l) + \mu_h \Pi_h (u'_l, u'_h) \leq \mu_l (v_l - \bar{u}_l) + \mu_h \Pi_h (\bar{u}_l, \bar{u}_h) = \mu_l (1 - \pi) (v_l - c_l). \]

When \(u'_l \leq \bar{u}_l\), the partial derivative of \(\Pi (u'_l, u'_h)\) with respect to \(u'_l\) is

\[
- \mu_l (1 - \pi + \pi F_l (u'_l)) + \mu_l \pi f_l (u'_l) (v_l - u'_l) + \mu_h \frac{v_h - c_h}{c_h - c_l} \\
- \mu_l (1 - \pi + \pi F_l (u'_l)) + \mu_l \pi f_l (u'_l) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (u'_l)) \frac{v_h - c_h}{c_h - c_l} = 0.
\]

Thus, for a given value of \(u'_h\), we must have

\[ \Pi (u'_l, u'_h) \leq \Pi (\bar{u}_l, u'_h) < \Pi (\bar{u}_l, \bar{u}_h) = \mu_l (1 - \pi) (v_l - c_l) \]

where the last inequality follows from the fact that \(\Pi_h\) is decreasing in \(u'_h\). Thus, such global deviations are unprofitable.

2. Consider next deviations with \(u'_h \in [c_h, \bar{u}_h]\). In this case, there must exist \(\bar{u}_l\) such that \(u'_h = U_h (\bar{u}_l)\) and thus \(F_h (u'_h) = F_l (\bar{u}_l)\). We can thus write the profits obtained from the deviation menu \((u'_l, u'_h)\) as

\[ \mu_l (1 - \pi + \pi F_l (u'_l)) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (\bar{u}_l)) \Pi_h (u'_l, u'_h). \]

(54)

We show that the function defined by (54) is strictly concave in \(u'_l\) for values of \(u'_l \in \text{Supp} (F_l)\) and decreasing for values of \(u'_l > \bar{u}_l\) so that this function is maximized at the value of \(u'_l\), which equates its partial derivative with zero. By (21), this partial derivative is zero when evaluated at \(u'_l = \bar{u}_l\), which completes the proof.

Note that for \(u'_l \in \text{Supp} (F_l)\), since \(\Pi_h\) is linear in \(u'_l\), the second derivative of (54) with respect to \(u'_l\) is given by

\[
\frac{\delta^2}{\delta (u'_l)^2} \mu_l (1 - \pi + \pi F_l (u'_l)) (v_l - u'_l). 
\]
Using the form of the distribution $F_1$ given by (48), we may rewrite this second derivative as

$$
\frac{\partial^2}{\partial (u'_i)^2} \mu_l \left( 1 - \pi + \pi F_1 \left( u'_i \right) \right) (v_l - u'_i) = \frac{\partial^2}{\partial (u'_i)^2} \mu_l \left( 1 - \pi \right) (v_l - c_l)^{\phi_l} (v_l - u'_i)^{1-\phi_l} = (\phi_l - 1) \phi_l \mu_l \left( 1 - \pi \right) (v_l - c_l)^{\phi_l} (v_l - u'_i)^{-1-\phi_l} < 0
$$

so that (54) is strictly concave in $u'_i$ for values of $u'_i \in \text{Supp} \, (F_1)$. For values $u'_i > \bar{u}_l$, $1 - \pi + \pi F_1 \left( u'_i \right) = 1$ and thus (54) satisfies

$$
\mu_l \left( v_l - u'_i \right) + \mu_h \left( 1 - \pi + \pi F_1 \left( \bar{u}_l \right) \right) \Pi_h \left( u'_i, u'_h \right).
$$

The derivative of this function with respect to $u'_i$ is given by

$$
-\mu_l + \mu_h \left( 1 - \pi + \pi F_1 \left( \bar{u}_l \right) \right) \frac{v_h - c_h}{c_h - c_l} < -\mu_l + \mu_h \frac{v_h - c_h}{c_h - c_l} = -\mu_l \phi_l < 0.
$$

Therefore, (54) is maximized at a value of $u'_i$, which equates the partial derivative of (54) with zero. This value must satisfy

$$
-\mu_l \left( 1 - \pi + \pi F_1 \left( u'_i \right) \right) + \mu_l \pi f_l \left( u'_i \right) \left( v_l - u'_i \right) + \mu_h \left( 1 - \pi + \pi F_1 \left( \bar{u}_l \right) \right) \frac{v_h - c_h}{c_h - c_l} = 0.
$$

Note that since (54) is strictly concave, at most one $u'_i$ exists that satisfies the above. The differential equation (21) implies that $u'_i = \bar{u}_l$ is a solution to the above equation. This implies that (54) must be maximized at $u'_i = \bar{u}_l$.

### A.2.2 Proofs of Propositions 3 and 4

We prove these propositions together. To begin, let $\phi_1$ be the value of $\phi_1$ that satisfies

$$
c_h \geq v_l + \frac{\pi (1 - \mu_l) (v_h - v_l)}{(1 - \pi) \left( \pi^{1-\phi_1} \phi_l - 1 \right)} \tag{55}
$$

with equality. Similarly, let $\phi_2$ be the value of $\phi_1$ that satisfies

$$
1 - \pi \geq \frac{\mu_h v_h + \mu_l v_l - v_l}{(1 - \phi_1)(\mu_h v_h + \mu_l v_l - c_h)} \tag{56}
$$

with equality. We first argue that (55) represents a lower bound on $\phi_1$ and (56) represents an upper bound on $\phi_1$ which lies below the lower bound defined by (55). In other words, the inequalities (55) and (56) partition the set $(-\infty, 0]$. We then prove that the equilibrium described in Proposition 4 exists—that is, in each case, no profitable local or global deviations exist when buyers use the equilibrium strategies defined jointly by Propositions 3 and 4.

**Lemma 12.** (55) is satisfied if and only if $\phi_1 \leq \phi_1 < 0$ and (56) is satisfied if and only if $\phi_1 \leq \phi_2$. Moreover, $\phi_2 < \phi_1 < 0$.

**Proof.** First, note that equation (55), which implicitly determines the threshold $\phi_1$, can be rewritten as

$$
(1 - \pi) \frac{1}{\phi_l} \geq \frac{\pi}{1 - \pi} \frac{v_h - v_l}{c_h - v_l} \mu_h + 1, \tag{57}
$$

...
or, after taking logs and substituting for \( \phi_l \), can be rewritten as

\[
\frac{\mu_h (v_h - c_h)}{c_h - c_l - \mu_h (v_h - c_l)} \log(1 - \pi) - \log(\mu_h \pi (v_h - v_l) + (1 - \pi) (c_h - v_l)) - \log[(1 - \pi) (c_h - v_l)] \geq 0.
\]

(58)

We show that the left side of (58) is a decreasing function of \( \mu_h \), that (58) is strictly satisfied when \( \mu_h = 0 \), and that (58) is weakly violated when \( \mu_h = 1 \). Hence, there is a unique threshold \( \mu_1 \) (and implied threshold \( \phi_1 \)) such that for all \( \mu_h \leq \mu_1 \) such that \( \phi_1 < 0 \), the separating condition (55) is satisfied. Differentiating the left side of (58) with respect to \( \mu_h \), we obtain

\[
\log(1 - \pi) \left( \frac{(v_h - c_h) (c_h - c_l)}{(c_h - c_l - \mu_h (v_h - c_l))^2} - \frac{\pi (v_h - v_l)}{\mu_h \pi (v_h - v_l) + (1 - \pi) (c_h - v_l)} \right),
\]

which is negative for all \( \pi \leq 1 \). Next, as \( \phi_1 \to 0 \) from below, it is immediate that (57) is satisfied since the left-hand side tends to infinity. As \( \mu_h \to 1 \), the term \((1 - \phi_1) / \phi_1 \to -1 \) and so (57) tends to the requirement that

\[
1 \geq \frac{v_h - v_l}{c_h - v_l} + (1 - \pi),
\]

which is violated since \( c_h < v_h \).

Next, consider equation (56), which implicitly determines the threshold \( \phi_2 \). Substituting for \( \phi_l \), one can show the inequality (56) is equivalent to

\[
\mu_h (v_h - v_l) \left[ 1 + (1 - \pi) \frac{v_h - c_h}{c_h - c_l} \right] \geq v_h - v_l + (c_h - v_l) (1 - \pi) \frac{v_h - c_h}{c_h - c_l}.
\]

(59)

Clearly, (59) represents a lower bound on \( \mu_h \), or, equivalently, an upper bound on \( \phi_1 \). Note that this equation is necessarily satisfied at \( \mu_h = 1 \). It is immediate that when \( \mu_h \) is such that \( \phi_1 = 0 \), equation (56) is violated since \( c_h > v_l \).

We now establish that \( \phi_2 < \phi_1 \) by proving that \( \phi_1 \leq \phi_2 \) implies \( \phi_1 < \phi_1 \). Suppose \( \phi_1 \leq \phi_2 \) and let \( \bar{v} = \mu_h v_h + \mu_l v_l \), so that we can write (56) as

\[
1 - \pi \geq \frac{\bar{v} - v_l}{(1 - \phi_l) (\bar{v} - c_h)}.
\]

(60)

Below, we will use the fact that (60) implies

\[
1 - \phi_l \geq \frac{\bar{v} - v_l}{(\bar{v} - c_h) (1 - \pi)} > \frac{\bar{v} - v_l}{\bar{v} - c_h} \Rightarrow -\phi_l > \frac{c_h - v_l}{\bar{v} - c_h}.
\]

To prove that (55) is violated when \( \phi_1 \leq \phi_2 \), note that (55) can be rearranged as

\[
(1 - \pi) \left[ (1 - \pi) \frac{1 - \phi_1}{\pi} - 1 \right] (c_h - v_l) - \pi \mu_h (v_h - v_l) \geq 0
\]

which can be simplified to

\[
(1 - \pi) (\bar{v} - c_h) + (1 - \pi) \frac{1}{\pi} (c_h - v_l) \geq \bar{v} - v_l.
\]

(61)

We will show that (61) is violated if (60) holds. Towardsthis end, define a function

\[
H(\pi) = (1 - \pi) (\bar{v} - c_h) + (1 - \pi) \frac{1}{\pi} (c_h - v_l)
\]

so that we must show \( H(\pi) < \bar{v} - v_l \). We argue that \( H(\cdot) \) is a strictly convex function which is decreasing.
at $\pi = 0$ and that, if $\pi$ satisfies (60), then $H(\pi) < H(0) = \bar{v} - v_l$, which completes the proof.

First, note that $H'(\cdot)$ is strictly convex since $\phi_1 < 0$ and

$$H'(\pi) = - (\bar{v} - c_h) - \frac{1}{\phi_1} (1 - \pi) \frac{1}{\phi_1} (c_h - v_l),$$

$$H''(\pi) = \frac{1}{\phi_1} \left( \frac{1}{\phi_1} - 1 \right) (1 - \pi) \frac{1}{\phi_1} - 2 (c_h - v_l) > 0.$$ 

Next, observe that $H(0) = \bar{v} - v_l$, $H'(0) \leq 0$ when $-\phi_1 \geq (c_h - v_l) / (\bar{v} - c_h)$ and $\lim_{\pi \rightarrow 1} H(\pi) = \infty$. Thus, there is a unique value $\pi^* > 0$ such that for all $\pi < \pi^*$, $H(\pi) \leq \bar{v} - v_l$.

Next, let $\hat{\pi}$ denote the value of $\pi$ such that (60) is satisfied with equality. We will prove that $H(\hat{\pi}) < \bar{v} - v_l$, so that $H(\pi) < \bar{v} - v_l$ for all $\pi \leq \hat{\pi}$.

Using the expression for $H(\pi)$, we have

$$H(\hat{\pi}) = \frac{\bar{v} - v_l}{(1 - \phi_1)(\bar{v} - c_h)} (\bar{v} - c_h) + \left( \frac{\bar{v} - v_l}{(1 - \phi_1)(\bar{v} - c_h)} \right)^{\frac{1}{\phi_1}} (c_h - v_l). \quad (62)$$

Straightforward algebra can be applied to (62) to show that $H(\hat{\pi}) < \bar{v} - v_l$ if and only if

$$\left( \frac{c_h - v_l}{\bar{v} - c_h} \right)^{\phi_1} \left( \frac{\bar{v} - v_l}{\bar{v} - c_h} \right)^{1 - \phi_1} > (-\phi_1 \phi_1 (1 - \phi_1)^{1 - \phi_1}. \quad (63)$$

Since $(\bar{v} - v_l) / (\bar{v} - c_h) = 1 + (c_h - v_l) / (\bar{v} - c_h)$, if we let $B(x) = x^{\phi_1}(1 + x)^{1 - \phi_1}$, then (63) can be written as the requirement that

$$B \left( \frac{c_h - v_l}{\bar{v} - c_h} \right) > B \left( -\phi_1 \right).$$

It is straightforward to show that $B'(x) < 0$ when $0 < x < -\phi_1$, and since (60) implies $-\phi_1 > (c_h - v_l) / (\bar{v} - c_h)$, (63) must hold. Consequently, $H(\pi) < H(\hat{\pi}) < \bar{v} - v_l$, which proves that $\phi_1 > \phi_2$.

**Definition of the Threshold, $\hat{u}_1$.** To prove Propositions 3 and 4, we first define the threshold $\hat{u}_1$ for various values of $\phi_1 < 0$.

**Case 1: $\phi_1 \leq \phi_2$.** The threshold satisfies $\hat{u}_1 = \bar{u}_1$, the upper bound of $F_1$, where $\bar{u}_1$ satisfies

$$\bar{v} - \bar{u}_1 = (1 - \pi)(\bar{v} - c_h). \quad (64)$$

**Case 2: $\phi_2 < \phi_1 < \phi_1$.** The threshold satisfies

$$v_l + (\hat{u}_1 - v_l) [1 - \pi + \pi F_1(\hat{u}_1)] \frac{1}{\phi_1} = \bar{v} - (1 - \pi)(\bar{v} - c_h) \quad (65)$$

where $F_1(\hat{u}_1)$ satisfies (23). As we will see below, in this case, the threshold will be such that $F_1(\hat{u}_1) \in (0, 1)$ so that the equilibrium is indeed mixed.

**Case 3: $\phi_1 < \phi_1 < 0$.** The threshold is any value such that $\hat{u}_1 < \bar{u}_1$, where the lower bound of the support of $F_1$ satisfies

$$(1 - \pi) \left[ \mu_l(v_l - \bar{u}_1) + \mu_c \Pi_l(v_l, c_h) \right] = \bar{v} - \left[ v_l + (1 - \pi) \frac{1}{\phi_1} (\bar{u}_1 - v_l) \right]. \quad (66)$$

This equation determines the lower bound as the value that equates profits from the worst (separating) menu and the best (pooling) menu where the best menu is determined as the value of $u_l$ such that $F_1(u_l) = 1$ when $F_1$ is determined by (21).

We now prove that the conjectured equilibria defined implicitly by the thresholds above, in each case, are indeed equilibria.
Lemma 13. Suppose $\phi_1 \leq \phi_1 < 0$. There exists an equilibrium with only separating menus.

Proof. It suffices to ensure that global deviations are unprofitable for buyers since, by construction, the distribution $F_1(u_1)$ ensures no local deviations are profitable. To rule out global deviations, a proof similar to that of Proposition 2 can be used. We show that for a given value of $u'_h$, the profit function is strictly concave in $u'_l$ and, therefore, it must be maximized at $u'_l = U^{-1}_h(u'_h)$, since at this value the derivative of the profit function is equal to zero (by construction).

Profits from such a local deviation are given by

$$
\mu_1 (1 - \pi + \pi F_1(u'_l)) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_h(u'_h)) \Pi_h(u'_l, u'_h).
$$

Since $\Pi_h$ is linear in $u'_l$, the second derivative of the above function is equal to the second derivative of profits from 1 type sellers. Using (21), we know that $(1 - \pi + \pi F_1(u'_l)) = \kappa (u'_l - v_l)^{-\phi_1}$ for some non-negative constant $\kappa$. Therefore, we have

$$
\frac{\partial^2}{\partial (u'_l)^2} \mu_1 (1 - \pi + \pi F_1(u'_l)) (v_l - u'_l) = -\mu_1 \kappa \frac{\partial^2}{\partial (u'_l)^2} (u'_l - v_l)^{1-\phi_1}
= -\mu_1 \kappa (1-\phi_1) (u'_l - v_l)^{-2-\phi_1} < 0.
$$

Lemma 14. Suppose $\phi_1 \leq \phi_2$. There exists an equilibrium with only pooling menus.

Proof. We first prove that no local deviations in the pooling equilibrium strictly improve profits. Below we demonstrate global deviations are also unprofitable. Recall that in an equilibrium with only pooling menus, the distribution $F_1(u_1)$ satisfies

$$
(1 - \pi + \pi F_1(u_1)) (\bar{v} - u_1) = (1 - \pi) (\bar{v} - c_h)
$$

where $\bar{v} = \mu_h v_h + \mu_1 v_1$, $U_h(u_1) = u_1$, $F_h(u_1) = F_1(u_1)$, and $\text{Supp}(F_1) = [c_h, \bar{v} - (1 - \pi) (\bar{v} - c_h)]$. Fix any utility, $u_1$, interior to the support of $F_1$ and consider a local deviation to the menu $(u'_l, u'_h) = (u_1, u_1 + \varepsilon)$. Profits from such a local deviation satisfy

$$
\mu_1 (1 - \pi + \pi F_1(u_1)) (v_l - u_1) + \mu_h (1 - \pi + \pi F_1(u_1 + \varepsilon)) \Pi_h(u_1, u_1 + \varepsilon)
= \mu_1 (1 - \pi + \pi F_1(u_1)) (v_l - u_1) + \mu_h (1 - \pi + \pi F_1(u_1 + \varepsilon)) \left[ v_h - u_1 - \varepsilon \frac{v_h - c_1}{c_h - c_1} \right].
$$

If local deviations are unprofitable, this function must be maximized at $\varepsilon = 0$, so that $F_1$ must satisfy

$$
\mu_h \pi f_1 (u_1) [v_h - u_1] - \mu_h (1 - \pi + \pi F_1(u_1)) \frac{v_h - c_1}{c_h - c_1} \leq 0.
$$

Totally differentiating (67) yields the following relationship between $F_1$ and $f_1$,

$$
\pi f_1(u_1) (\bar{v} - u_1) = (1 - \pi + \pi F_1(u_1))
$$

so that local deviations are unprofitable if

$$
\mu_h \pi f_1 (u_1) [v_h - u_1] - \mu_h \pi f_1 (u_1) (\bar{v} - u_1) \frac{v_h - c_1}{c_h - c_1} \leq 0.
$$

Since $F_1$ is continuous in our constructed equilibrium, we may simplify this condition using straightforward algebra as

$$
u_1 (v_h - c_h) \leq \bar{v} (v_h - c_1) - v_h (c_h - c_1).
$$
Consequently, we see that it suffices to check that this deviation is unprofitable at \( \max \text{Supp}(F_l) \). Using \( u_t = \bar{v} - (1 - \pi) (\bar{v} - c_h) \), simple algebraic manipulations show that this local deviation is unprofitable as long as

\[
\frac{\bar{v} - v_t}{(1 - \phi_l) (\bar{v} - c_h)} \leq 1 - \pi,
\]

which is guaranteed by Lemma 12 since \( \phi_1 \leq \phi_2 \).

To rule out global deviations, we show that for any value of \( u_h \in \text{Supp}(F_l) \), the profit function in increasing in \( u_t \) for all \( u_t \leq u_h \). Thus, profits are maximized at the pooling menu \( u_t = u_h \) so that there are no profitable deviations.

Profits associated with any global deviation \((u_t', u_h')\) with \( u_t' \leq u_h' \) and \( u_h' \in \text{Supp}(F_l) \) are given by

\[
\mu_l \left( 1 - \pi + \pi F_l(u_t') \right) (v_t - u_t') + \mu_h \left( 1 - \pi + \pi F_l(u_h') \right) \Pi_h(u_t', u_h').
\]

Differentiating, we obtain

\[
\begin{align*}
\mu_l \pi f_l(u_t') (v_t - u_t') - \mu_l \left( 1 - \pi + \pi F_l(u_t') \right) &+ \mu_h \left( 1 - \pi + \pi F_l(u_h') \right) \frac{v_h - c_h}{c_h - c_l} \\
\mu_l \pi f_l(u_t') (v_t - u_t') - \mu_l \left( 1 - \pi + \pi F_l(u_t') \right) &+ \mu_h \left( 1 - \pi + \pi F_l(u_h') \right) \frac{v_h - c_h}{c_h - c_l} = \\
\mu_l \pi f_l(u_t') (v_t - u_t') - \mu_l \pi f_l(u_h') (v_t - u_h') &- \mu_l \phi_l \left( 1 - \pi + \pi F_l(u_t') \right)
\end{align*}
\]

(70)

where the inequality follows from the fact that \( u_t' \leq u_h' \) so that \( F_l(u_h') \geq F_l(u_t') \). Using (68) to substitute for \( \pi f_l(u_t') \), we can write the last line of (70) as

\[
\mu_l \left( 1 - \pi + \pi F_l(u_t') \right) \left[ 1 + \frac{v_t - \bar{v}}{v_t - u_t'} - \phi_l \right].
\]

Since \( u_t' \leq u_h' \leq \max \text{Supp}(F_l) \), the expression in brackets takes its minimum value at \( u_t' = \max \text{Supp}(F_l) \) so that

\[
1 + \frac{v_t - \bar{v}}{v_t - u_t'} - \phi_l \geq 1 + \frac{v_t - \bar{v}}{(1 - \pi) (\bar{v} - c_h)} - \phi_l \geq 0
\]

where the second inequality follows from (69). This implies that the expression in (70) is positive so that profits are globally maximized at \( u_t' = u_h' \) for all \( u_t' \in \text{Supp}(F_l) \).

**Lemma 15.** Suppose \( \phi_2 < \phi_1 < \phi_1 \). There exists a mixed equilibrium.

**Proof.** Recall that the threshold \( \hat{u}_1 \) is such that the constructed equilibrium features pooling contracts for \( u_t \in [\min \text{Supp}(F_l), \hat{u}_1] \) and separating menus for \( u_t \in (\hat{u}_1, \max \text{Supp}(F_l)) \). First, we claim that when \( \phi_2 < \phi_1 < \phi_1 \), then \( \hat{u}_1 \) is interior in the sense that \( c_h < \hat{u}_1 < \bar{u}(\hat{u}_1) \). Second, we prove that no local or global deviations are profitable.

To see that \( \hat{u}_1 \) is interior, conjecture that \( \hat{u}_1 > c_h \) (we will verify it later), in which case \( \hat{u}_1 \) must satisfy

\[
\bar{v} - \left( v_t + (\hat{u}_1 - v_t) \left[ (1 - \pi) \frac{\bar{v} - c_h}{\bar{v} - \hat{u}_1} \right] ^{\frac{1}{\phi_l}} \right) - (1 - \pi) (\bar{v} - c_h) = 0.
\]

(71)

Let \( H(\hat{u}_1) \) denote the left-hand side of (71). We will prove that when \( \phi_2 < \phi_1 < \phi_1 \), there are two solutions to \( H(\hat{u}_1) = 0 \) with \( \hat{u}_1 > c_h \).

First, observe that one solution to \( H(\hat{u}_1) = 0 \) is given by

\[
\hat{u}_1 = \bar{u} = \bar{v} - (1 - \pi) (\bar{v} - c_h).
\]

\[\text{Recall that equilibrium profits satisfy } \Pi = (1 - \pi)(\bar{v} - c_h) \text{ when the worst menu offered in equilibrium is the pooling, monopsony menu.}\]
This solution coincides with the conjecture that all menus are pooling and therefore \( \hat{u} (\hat{u}_1) = \hat{u}_1 \).

We argue that there exists another solution \( \hat{u}_1 \in \{c_h, \bar{u}\} \). We show this by proving that \( H(\cdot) \) is convex, \( H(c_h) > 0 \), and \( H'(\bar{u}) > 0 \) so that an additional solution in the interval \( \{c_h, \bar{u}\} \) must exist.

Note that

\[
H'(u) = -\left[ (1-\pi) \frac{\bar{v} - c_h}{\bar{v} - u} \right]^{\frac{1}{\phi_1}} - (u - v_1) \frac{1}{\phi_1} \left[ (1-\pi) \frac{\bar{v} - c_h}{\bar{v} - u} \right]^{\frac{1}{\phi_1}-1} (1-\pi) (\bar{v} - c_h) (\bar{v} - u)^{-2}.
\]

By differentiating \( H'(\cdot) \) and applying extensive algebraic manipulations (available upon request), one can show that \( H''(\cdot) \geq 0 \). Recall that \( \bar{u} \) is defined so that \( H(\bar{u}) = 0 \) and

\[
H'(\bar{u}) = -1 - \frac{1}{\phi_1} \frac{\bar{u} - v_1}{\bar{v} - \bar{u}} = H'(\bar{u}) = \frac{1}{\phi_1} \left[ 1 - \frac{\bar{v} - v_1}{1 - \phi_1 (1 - \phi_1) (\bar{v} - c_h)} \right]
\]

where the second equality is obtained by substituting for \( \bar{u} \) and rearranging terms. When \( \phi_1 > \phi_2 \), the term in brackets is negative, by Lemma 12, so that \( H'(\bar{u}) > 0 \). Finally, one can show that \( H(c_h) \) satisfies

\[
H(c_h) = \frac{1}{(1-\pi) \frac{\bar{v} - c_h}{\bar{v} - v_1} - (1 - \pi)} \left[ v_1 + \frac{\pi (\bar{v} - v_1)}{(1-\pi) \frac{\bar{v} - c_h}{\bar{v} - v_1} - (1 - \pi)} - c_h \right].
\]

From Lemma 12, since \( \phi_1 < \phi_2 \), the term in brackets is strictly positive, and, since the leading fraction is also positive, we must have \( H(c_h) > 0 \).

Hence, when \( \phi_2 < \phi_1 < \phi_1 \), there must exist a solution to \( H(\hat{u}_1) = 0 \) with \( \hat{u}_1 \in \{c_h, \bar{u}\} \). When \( \hat{u}_1 < \bar{u} \), one can show that \( F_1(\hat{u}_1) < 1 \) when \( F_1 \) is determined by (23) on the interval \( [c_h, \hat{u}_1] \), which confirms the conjecture that \( \hat{u}_1 \) is the interior of the support of \( F_1 \).

We now show that buyers cannot improve their profits by deviating from the constructed mixed allocation. As in Lemma 13 with only separation, the distribution \( F_1 \) for \( u_t \in [\hat{u}_1, \max \text{Supp}(F_1)] \) is chosen to ensure local deviations are not profitable. It remains to show, then, that local deviations are not profitable in the pooling region and that no global deviations are profitable. As in Lemma 14 with only pooling menus, it suffices to ensure that at the upper bound of the pooling region, \( \hat{u}_1 \), no local deviations are profitable, or

\[
\hat{u}_1 (v_h - c_h) \leq \bar{v} (v_h - c_1) - v_h (c_h - c_1) \quad (72)
\]

To prove that (72) holds, first note that since \( \phi_2 < \phi_1 < \phi_1 \), we have \( c_h < \hat{u}_1 < \bar{u} (\hat{u}_1) \). We now prove that (72) is satisfied at \( \hat{u}_1 \). Algebra (available upon request) shows that (72) may be written as

\[
\hat{u}_1 \leq \frac{-\phi_1}{1 - \phi_1} \bar{v} + \frac{1}{1 - \phi_1} v_1.
\]

Since \( H(\hat{u}_1) = 0 \), if \( H\left( \frac{-\phi_1}{1 - \phi_1} \bar{v} + \frac{1}{1 - \phi_1} v_1 \right) \leq 0 \) then since \( H(\cdot) \) is convex, (72) must be satisfied.

Using the form of \( H(\cdot) \) implied by the left-hand side of (71), one can show that

\[
H\left( \frac{-\phi_1}{1 - \phi_1} \bar{v} + \frac{1}{1 - \phi_1} v_1 \right) = (\bar{v} - v_1) \left[ \frac{\bar{v} - v_1 - (1 - \pi) (\bar{v} - c_h)}{\bar{v} - v_1} + \phi_1 \left( 1 - \phi_1 \right) \frac{1}{\phi_1} \left( 1 - \pi \right) \left( \bar{v} - c_h \right) \left( \bar{v} - v_1 \right) \right].
\]

We now show that the term in brackets on the right side of (73) is negative. To simplify notation, define \( \xi = (1 - \pi) (\bar{v} - c_h) / (\bar{v} - v_1) \) so that the term in brackets can be written compactly as

\[
1 - \xi + \phi_1 (1 - \phi_1) \xi^{-1} \xi^{\frac{1}{\phi_1}}.
\]
Let $G(\xi) = 1 - \xi + \phi_1 (1 - \phi_1)^{\frac{1}{\phi_2}}$ and observe that for $\xi \leq 1/(1 - \phi_1)$, we have
$$G'(\xi) = -1 + [(1 - \phi_1) \xi]^\frac{1}{\phi_2} \geq 0$$
so that for low values of $\xi$, $G(\xi)$ is an increasing function.

Since $\phi_1 > \phi_2$, (60) implies that $\xi < 1/(1 - \phi_1)$. Moreover, since $G(1/(1 - \phi_1)) = 0$, it must be that $G(\xi) \leq G(1/(1 - \phi_1)) = 0$, which ensures the term in brackets in (73) is indeed negative as desired.

To rule out global deviations, one can use the arguments provided in the proofs of Lemmas 13 and 14 in each region of the Supp $(F_1)$. Since the arguments are exact replicas of the arguments above, we omit them here.

A.2.3 Proof of Theorem 2

We begin with a lemma which ensures that the marginal distribution $F_1$ is continuous (i.e., it has no mass points) when $\phi_1 \neq 0$. We then prove uniqueness of the equilibrium first for $\phi_1 > 0$ and then for $\phi_1 < 0$. (In Appendix C, we demonstrate uniqueness for $\phi_1 = 0$.)

Lemma 16. If $\phi_1 \neq 0$, then $F_1$ is continuous.

Proof. Recall from Lemma 11 that if $F_1$ has a mass point, then it occurs at $\hat{u}_1 = v_1$. As well, from Lemma 9, there must exist a positive measure set $S = \{\hat{u}_1, u_h\}$ such that each equilibrium menu $(\hat{u}_1, u_h)$ has $\Pi_1 = 0$. Let $u_h$ denote the lowest value of $u_h$ for which $(\hat{u}_1, u_h)$ belongs to the closure of the set $S$ and let $\bar{u}_h$ denote the highest such value. Without loss of generality, we may assume that $(\hat{u}_1, u_h)$ and $(\bar{u}_1, \bar{u}_h)$ belong to $S$ and thus deliver the same profits to a buyer as the equilibrium level of profits.

Consider then the value of two different deviations, $(\hat{u}_1 - \varepsilon, \bar{u}_h)$ and $(\hat{u}_1 + \varepsilon, \bar{u}_h)$, for a small value of $\varepsilon > 0$, both of which must be feasible. The profits from these deviations are given by

$$\Pi(\hat{u}_1 - \varepsilon, \bar{u}_h) = \mu_h (1 - \pi + \pi F_h \bar{u}_h) \Pi_h (\hat{u}_1 - \varepsilon, \bar{u}_h) + \mu_1 (1 - \pi + \pi F_1 (\hat{u}_1 - \varepsilon)) \varepsilon$$
$$\Pi(\hat{u}_1 + \varepsilon, \bar{u}_h) = \mu_h (1 - \pi + \pi F_h \bar{u}_h) \Pi_h (\hat{u}_1 + \varepsilon, \bar{u}_h) - \mu_1 (1 - \pi + \pi F_1 (\hat{u}_1 + \varepsilon)) \varepsilon.$$ 

These equalities are valid because $F_h$ does not have a mass point and $F_1$ does not have a mass point for $u_1 > v_1$ or $u_1 < v_1$. Since $F_1$ is then left or right differentiable at $\hat{u}_1$, we have that

$$\frac{d}{d\varepsilon} \Pi(\hat{u}_1 - \varepsilon, \bar{u}_h) \bigg|_{\varepsilon=0} = -\mu_h (1 - \pi + \pi F_h \bar{u}_h) \frac{v_h - c_h}{c_h - c_1} + \mu_1 (1 - \pi + \pi F_1^+ (\hat{u}_1))$$
$$\frac{d}{d\varepsilon} \Pi(\hat{u}_1 + \varepsilon, \bar{u}_h) \bigg|_{\varepsilon=0} = \mu_h (1 - \pi + \pi F_h \bar{u}_h) \frac{v_h - c_h}{c_h - c_1} - \mu_1 (1 - \pi + \pi F_1^- (\hat{u}_1)).$$ 

The optimality of menus in $S$ implies that the above expressions must both be non-positive. Since the equilibrium distributions are well-behaved above and below $v_1$, the equilibrium necessarily exhibits the strict rank-preserving property by Theorem 1 and therefore, $F_1^- (\hat{u}_1) = F_h (\bar{u}_h)$ and $F_1^+ (\hat{u}_1) = F_h (\bar{u}_h)$. As a result, the above inequalities imply that

$$-\mu_h \frac{v_h - c_h}{c_h - c_1} + \mu_1 \leq 0$$
$$\mu_h \frac{v_h - c_h}{c_h - c_1} - \mu_1 \leq 0.$$ 

When $\phi_1 \neq 0$, one of the above is violated. Hence, a profitable deviation exists yielding the necessary contradiction.
Case 1: $\phi_1 > 0$. As we have shown, any separating equilibrium is uniquely determined. Thus, in order to show the uniqueness of the equilibrium in this case, it remains to show that any equilibrium is separating. To see this, suppose to the contrary that $u_1 = u_h$ for some menu offered in equilibrium. Now, consider the following alternative menu $(u_1 - \epsilon, u_h)$ for a small and positive value of $\epsilon$. This menu is feasible and the change in the profits for a small value of $\epsilon$ is given by

$$
\mu_1(1 - \pi + \pi F_1(\epsilon_u))\epsilon - \mu_h(1 - \pi + \pi F_h(u_h)) \frac{v_h - c_h}{c_h - c_l} \epsilon - \mu_1 \pi f_1^-(v_1 - u_1)\epsilon
$$

where $f_1^-$ is the left-derivative of $F_1$ at $u_1$; recall from Appendix A.1.2 that $F_1$ must be differentiable.

Using the definition of $\phi_1$ and strict rank preserving property, we can write the above as

$$
\mu_1 \phi_1 (1 - \pi + \pi F_1(\epsilon_u)) \epsilon - \mu_1 \pi f_1^-(v_1 - u_1)\epsilon.
$$

The above expression must be positive: $\phi_1 > 0, F_1$ and $f_1^-(u_1)$ are weakly positive, and $u_1 > v_1$ since $u_1 = u_h \geq c_h > v_1$ where $c_h > v_1$ by the lemons assumption. Therefore, this alternative menu is a profitable deviation which yields the necessary contradiction.

Case 2: $\phi_1 < 0$. To prove the equilibrium characterized in Proposition 4 is unique, we first prove that in any equilibrium with $\phi_1 < 0$, if $\bar{u} = \max \text{Supp}(F_1)$, then $U_h(\bar{u}) = \bar{u}$ so that the best menu in equilibrium is a pooling menu. Next, we prove that if the equilibrium has a pooling region, the region begins at the lower bound of the support of $F_1$ or ends at the upper bound of $F_1$. Additionally, if the equilibrium features a separating region, this region must end at the upper bound of the support of $F_1$. These results imply that any equilibrium must take one of the three forms described in Proposition 4: only separating, only pooling, or mixed. Finally, we show that the necessary conditions for each type of equilibrium to exist are mutually exclusive so that at most one type of equilibrium exists for each region of the parameter space, ensuring our equilibrium is unique for all $\phi_1 < 0$. We prove these results in the following sequence of lemmas.

Lemma 17. If $\phi_1 < 0$, then the best equilibrium menu is a pooling menu.

Proof. Let $\bar{u} = \max \text{Supp}(F_1)$ and suppose for contradiction that $U_h(\bar{u}) > \bar{u}$. Consider a deviation menu with $(u'_1, u'_h) = (\bar{u} + \epsilon, U_h(\bar{u}))$. Since $U_h(\bar{u}) > \bar{u}$, this menu is incentive compatible and has $F_1(u'_1) = F_1(u'_h) = 1$. This menu increases the buyer’s profits relative to the menu $(\bar{u}, U_h(\bar{u}))$ by the amount

$$
-\mu_1 \epsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} = -\mu_1 \phi_1 \epsilon > 0
$$

where the inequality follows from $\phi_1 < 0$. This profitable deviation yields the necessary contradiction so that we must have $U_h(\bar{u}) = \bar{u}$. ■

Lemma 18. If $\phi_1 < 0$ and an equilibrium features $[u_1, u_2] \subseteq \text{Supp}(F_1)$ such that $U_h(u_1) = u_1$ for $u_1 \in [u_1, u_2]$, then either $u_1 = \min \text{Supp}(F_1)$ or $u_2 = \max \text{Supp}(F_1)$.

Proof. Suppose towards a contradiction that a pooling interval with $u_1 > \min \text{Supp}(F_1)$ and $u_2 < \max \text{Supp}(F_1)$ exists. Then there must exist intervals sufficiently close to and below $u_1$ and above $u_2$, respectively, in which the equilibrium menus feature separation. Since in these intervals, $U_h(u_1) > u_1$ but $U_h(u_1) = u_1$ and $U_h(u_2) = u_2$, we must have $\lim_{u_1 \to u_1} U_h(u_1) \leq 1$ and $\lim_{u_1 \to u_2} U_h(u_1) \geq 1$. In any region with $U_h(u_1) > u_1$, the distribution $F_1$ must also satisfy

$$
\frac{\pi f_1(u_1)}{1 - \pi + \pi F_1(u_1)} = \frac{-\phi_1}{u_1 - v_1}
$$
As the conjectured equilibrium satisfies equilibrium features separation in 
\[ \Phi_u = \Phi_{\bar{u}} \]
for some \( \bar{u} \) interval \( \Phi_u < \Phi_{\bar{u}} \).

Proof. Suppose by way of contradiction that an equilibrium features separation (\( \Phi_u < \Phi_{\bar{u}} \)) then rearranging terms, we obtain
\[ -\mu_1 \phi_1 u_1 - \mu_\Phi \Phi_u - c_l \Phi_u(u_1) = \Phi(1 - \pi + \pi \Phi_1(u_1))^{-1} \]
where \( \Phi \) denotes the level of equilibrium profits.

Using these features of the conjectured equilibrium, in the separating regions, \( U''_h(u_1) \) satisfies
\[ -\mu_1 \phi_1 - (1 - \mu_1 \phi_1) U''_h(u_1) = \frac{\Phi}{1 - \pi + \pi \Phi_1(u_1)} \frac{\phi_1}{u_1 - v_1} \]
and so \( U''_h(u_1) \) satisfies
\[ - (1 - \mu_1 \phi_1) U''_h(u_1) = \frac{\Phi \Phi_1(u_1)}{1 - \pi + \pi \Phi_1(u_1)} \frac{\phi_1}{u_1 - v_1} + \frac{\Phi}{1 - \pi + \pi \Phi_1(u_1)} \frac{-\phi_1}{u_1 - v_1} \]
which implies that \( U_h \) is concave when \( \phi_1 < 0 \). However, the existence of the pooling region implies that \( U''_h(u_2) \) \( \Phi_u \Phi_{\bar{u}} \) \( \Phi_u(u_1) \), which contradicts the concavity of \( U_h \) given that \( u_1 < u_2 \). Hence, either \( u_1 = \min \text{Supp}(F_1) \) or \( u_2 = \max \text{Supp}(F_1) \).

Lemma 19. If \( \phi_1 < 0 \) and an equilibrium features \( [u_1, u_2] \subseteq \text{Supp}(F_1) \) such that \( U_h(u_1) > u_1 \) for \( u_1 \in (u_1, u_2) \), then \( u_2 = \max \text{Supp}(F_1) \).

Proof. Suppose by way of contradiction that an equilibrium features separation (\( U_h(u_1) > u_1 \)) on an interval \( [u_1, u_2] \subseteq \text{Supp}(F_1) \) with \( u_2 < \max \text{Supp}(F_1) \). Then there must exist a pooling interval \( [u_2, \bar{u}] \) for some \( \bar{u} \). Since \( u_2 > \min \text{Supp}(F_1) \), Lemma 18 implies that \( \bar{u} = \max \text{Supp}(F_1) \). Since the conjectured equilibrium features separation in \( [u_1, u_2] \) with \( U_h(u_1) \rightarrow u_1 \) as \( u_1 \rightarrow u_2 \), we must have \( U''_h(u_2) \leq 1 \). As the conjectured equilibrium satisfies
\[ \frac{\Phi \Phi_1(u_1)}{1 - \pi + \pi \Phi_1(u_1)} = \frac{-\phi_1}{u_1 - v_1} \]
on the interval \( [u_1, u_2] \), \( U''_h(u_2) \leq 1 \) implies
\[ \frac{1}{1 - \mu_1 \phi_1} \left[ -\mu_1 \phi_1 + \frac{\Phi}{1 - \pi + \pi \Phi_1(u_2)} \frac{-\phi_1}{u_2 - v_1} \right] \leq 1 \]
or
\[ -\phi_1 \Phi \leq (1 - \pi + \pi \Phi_1(u_2)) (u_2 - v_1). \]
Since \( u_2 < \bar{u} \), \( \Phi(u_2) < 1 \) so that
\[ -\phi_1 \Phi < u_2 - v_1. \] (74)
Moreover, Lemma 17 ensures that the best equilibrium menu is pooling with utility \( \bar{u} \) and, therefore, equilibrium profits satisfy \( \Phi = \bar{v} - \bar{u} \). Using the fact that \( u_2 < \bar{u} \), substituting for \( \Phi \) in (74), and rearranging terms, we obtain
\[ 0 < \phi_1 - \frac{v_1 - \bar{u}}{\bar{v} - \bar{u}}. \] (75)

We will show that (75) implies that a cream-skimming deviation must be a profitable deviation from the best (pooling) menu, yielding the necessary contradiction. Since the conjectured equilibrium features pooling in the interval \( [u_2, \bar{u}] \), for \( u_1 \) in this interval, the equilibrium satisfies
\[ (1 - \pi + \pi \Phi_1(u_1))(\bar{v} - u_1) = (1 - \pi)(\bar{v} - \bar{u}) \]
so that
\[
f_1(u_1) = \frac{1 - \pi + \pi F_1(u_1)}{\pi(\bar{v} - u_1)}. \tag{76}\]

Consider then a cream-skimming deviation of the form \((u'_1, u'_h) = (\bar{u} - \varepsilon, \bar{u})\), which yields profits equal to
\[
(1 - \pi + \pi F_1(\bar{u} - \varepsilon))u_1(v_1 - \bar{u} + \varepsilon) + (1 - \pi + \pi F_1(\bar{u}))u_h \Pi_h(\bar{u} - \varepsilon, \bar{u}). \tag{77}
\]
Differentiating (77) with respect to \(\varepsilon\) and evaluating it at \(\varepsilon = 0\), we obtain
\[
(1 - \pi + \pi F_1(\bar{u}))\mu_1(1 - \bar{u}) - (1 - \pi + \pi F_1(\bar{u}))\mu_h \frac{v_h - c_h}{c_h - c_1}
\]
which, given that \(F_1(\bar{u}) = 1\) and \(f_1(\bar{u}) = 1/|\pi(\bar{v} - \bar{u})|\), can be written as
\[
\mu_1 \left[ \phi_1 - \frac{v_1 - \bar{u}}{\bar{v} - \bar{u}} \right] > 0, \tag{78}\]
where the inequality follows from (75). Hence, this cream-skimming deviation strictly increases the buyers’ profits relative to the conjectured equilibrium level, a contradiction.

Since the only possible equilibria when \(\phi_1 < 0\), then, are fully separating (except at the upper bound of the support of \(F_1\)), fully pooling, or mixed, we need only prove that only one of these equilibria may exist for any value of \(\phi_1\). We have already shown in the proof of Proposition 4 that \(\phi_2 < \phi_1 < 0\). Recall that a necessary condition for a fully pooling equilibrium is that \(\phi_1 \leq \phi_2\). Hence, there is no fully pooling equilibrium when \(\phi_1 > \phi_2\). Similarly, a necessary condition for a fully separating equilibrium is that \(\phi_1 \geq \phi_1\) so that when \(\phi_1 < \phi_1\), no fully separating equilibrium exists. This means that in the interval \(\phi_2 < \phi_1 < \phi_1\), the only possible equilibrium is a mixed equilibrium. Moreover, the threshold in the mixed equilibrium is interior to the support of \(F_1\) only if \(\phi_1\) lies between \(\phi_2\) and \(\phi_1\). Hence, at most one of these types of equilibria may exist for any value of \(\phi_1 < 0\), proving that the equilibrium described in Proposition 13 is unique.

A.3 Proofs from Section 5

A.3.1 Proof of Proposition 5

In a slight abuse of notation, we write welfare as a function of \(p\),
\[
W(p, \mu_h) = (1 - \mu_h)(v_1 - c_1) + \mu_h [(1 - p)X_1(p) + pX_2(p)}] \tag{79}\]
where
\[
X_n(p) = (v_h - c_h) \int_{c_1}^{\bar{u}(\pi(p))} x_h(u_t)(v_h - c_h) \, d(F_1^n(u_t, \pi(p)))
\]
and
\[
\pi(p) = \frac{2p}{1 + p}. \tag{80}\]

Several facts follow immediately from our characterization of equilibrium. First, note that \(X_1(0) = X_2(0) = 1\) when \(\phi_1 < 0\), which implies immediately that welfare is (weakly) maximized at \(p = \pi(0) = 0\) in this region of the parameter space. Second, note that \(X_1(0) = X_2(0) = 0\) when \(\phi_1 > 0\), while \(X_n(p) > 0\) for all \(p \in (0, 1]\). Hence, welfare is minimized at \(p = \pi(0) = 0\) in this region of the parameter space. To show that welfare is maximized at an interior value of \(\pi\) when \(\phi_1 > 0\), we will prove that \(\lim_{p \to 1} W_p(p, \mu_h) < 0\).
To this end, first note that
\[
\frac{1}{\mu_h} W_p(p, \mu_h) = X_2(p) - X_1(p) + pX'_2(p) + (1 - p)X'_1(p)
\]

In what follows, we will prove a sequence of results:

1. \(\lim_{p \to 1} X_2(p) - X_1(p) = 0\);
2. \(\lim_{p \to 1} (1 - p)X'_1(p) = 0\);
3. \(\lim_{p \to 1} pX'_2(p) < 0\).

The first result follows immediately from the fact that \(F(u)\) converges to a degenerate distribution at \(p = \pi(1) = 1\). To prove the second result, we first integrate \(X'_1(p)\) by parts:

\[
\begin{align*}
\frac{1}{\nu_h - c_h} (1 - p)X'_1(p) &= (1 - p) \frac{d}{dp} \left[ \bar{u} \left( \bar{u}_1(\pi(p)) \right) \right] - \int_{c_1}^{\bar{u} \left( \bar{u}_1(\pi(p)) \right)} x_h(\bar{u}_1(\pi(p))) \frac{dF_1(\bar{u}_1; \pi(p))}{d\bar{u}_1} d\bar{u}_1 \\
&= - (1 - p) \frac{d}{dp} \left[ \bar{u} \left( \bar{u}_1(\pi(p)) \right) \right] - \int_{c_1}^{\bar{u} \left( \bar{u}_1(\pi(p)) \right)} x_h(\bar{u}_1) \frac{dF_1(\bar{u}_1; \pi(p))}{d\bar{u}_1} d\bar{u}_1.
\end{align*}
\]

From the definition of \(F(u)\) in (48), we have

\[
\frac{dF_1(\bar{u}_1; \pi(p))}{d\bar{u}_1} = - \frac{F_1(\bar{u}_1; \pi)}{\pi(1 - \pi)}.
\]

Therefore,

\[
\frac{1}{\nu_h - c_h} (1 - p)X'_1(p) = (1 - p) \frac{d}{dp} \left[ \bar{u} \left( \bar{u}_1(\pi(p)) \right) \right] - \int_{c_1}^{\bar{u} \left( \bar{u}_1(\pi(p)) \right)} x_h(\bar{u}_1) \frac{dF_1(\bar{u}_1; \pi(p))}{d\bar{u}_1} d\bar{u}_1.
\]

Using (80), we obtain

\[
\frac{1}{\nu_h - c_h} (1 - p)X'_1(p) = \frac{2}{\pi(2 - \pi)} \frac{2}{(1 + p)^2} \left[ \bar{u} \left( \bar{u}_1(\pi(p)) \right) \right] - \int_{c_1}^{\bar{u} \left( \bar{u}_1(\pi(p)) \right)} x_h(\bar{u}_1) dF_1(\bar{u}_1; \pi(p))
\]

Since \(F_1\) becomes degenerate as \(\pi \to 1\) and \(\lim_{p \to 1} \pi = 1\), this final result implies

\[
\lim_{p \to 1} (1 - p)X'_1(p) = (\nu_h - c_h) \times 2 \times \frac{1}{2} \times 0 = 0.
\]

This completes the proof of the second claim above.
To prove the third result, we first integrate $X'_2(p)$ by parts and differentiate:

$$
\frac{1}{v_h - c_h} p X'_2(p) = p \frac{d}{dp} \left[ x_h(u_1) \int_{c_1}^{\bar{u}_1(\pi(p))} x_h(u_1) d \left( F^2_1(u_1; \pi(p)) \right) \right]
= p \frac{d}{dp} \left[ x_h(\bar{u}_1(\pi(p))) - \int_{c_1}^{\bar{u}_1(\pi(p))} x_h'(u_1) \left( F^2_1(u_1; \pi(p)) \right) du_1 \right]
= -p \int_{c_1}^{\bar{u}_1(\pi(p))} x_h'(u_1) \frac{dF^2_1(u_1; \pi(p))}{d \pi} \frac{d \pi}{du_1}.
$$

Since

$$
\frac{d}{d \pi} F^2_1(u_1; \pi) = 2F_1(u_1; \pi) \frac{dF_1(u_1; \pi(p))}{d \pi} = -\frac{2}{\pi(1 - \pi)} F^2_1(u_1; \pi),
$$
we have

$$
\frac{1}{v_h - c_h} p X'_2(p) = \frac{2p}{\pi(1 - \pi)} \frac{d \pi(p)}{d \pi} \int_{c_1}^{\bar{u}_1(\pi(p))} x'_h(u_1) F^2_1(u_1; \pi(p)) du_1
= \frac{2}{(2 - \pi)(1 - \pi)} \frac{2}{(1 + p)^2} \int_{c_1}^{\bar{u}_1(\pi(p))} x'_h(u_1) F^2_1(u_1; \pi(p)) du_1
= \frac{2}{(2 - \pi)(1 - \pi)} \frac{2}{(1 + p)^2} \left[ x_h(\bar{u}_1(\pi(p))) - \int_{c_1}^{\bar{u}_1(\pi(p))} x_h(u_1) d \left( F^2_1(u_1; \pi(p)) \right) \right].
$$

To prove the result, we will show that

$$
\lim_{\pi \to 1} \frac{1}{1 - \pi} \left[ x_h(\bar{u}_1(\pi)) - \int_{c_1}^{\bar{u}_1(\pi)} x_h(u_1) d \left( F^2_1(u_1; \pi) \right) \right] < 0.
$$

Define $H(\pi)$ as

$$
H(\pi) = x_h(\bar{u}_1(\pi)) - \int_{c_1}^{\bar{u}_1(\pi)} x_h(u_1) d \left( F^2_1(u_1; \pi) \right).
$$

Since $\lim_{\pi \to 1} H(\pi) = \lim_{\pi \to 1} 1 - \pi = 0$, we will apply L'Hopital's rule:

$$
\lim_{\pi \to 1} \frac{H(\pi)}{1 - \pi} = -\lim_{\pi \to 1} H'(\pi).
$$

Next, using integration by parts, we have

$$
H(\pi) = \int_{c_1}^{\bar{u}_1(\pi)} x_h'(u_1) F^2_1(u_1; \pi) du_1.
$$

Therefore, using (81), we have

$$
H'(\pi) = x'_h(\bar{u}_1(\pi)) \frac{d \bar{u}_1}{d \pi} + \int_{c_1}^{\bar{u}_1(\pi)} x_h'(u_1) \frac{dF^2_1(u_1; \pi)}{d \pi} du_1
= x'_h(\bar{u}_1(\pi)) \frac{d \bar{u}_1}{d \pi} - \frac{2}{\pi(1 - \pi)} \int_{c_1}^{\bar{u}_1(\pi)} x_h'(u_1) F^2_1(u_1; \pi) du_1
= x'_h(\bar{u}_1(\pi)) \frac{d \bar{u}_1}{d \pi} - \frac{2}{\pi(1 - \pi)} H(\pi).
$$
Therefore,
\[
\lim_{\pi \to 1} \frac{H(\pi)}{1 - \pi} = -\lim_{\pi \to 1} x_h'(\bar{u}_1(\pi)) \frac{d\bar{u}_1}{d\pi} + 2 \lim_{\pi \to 1} \frac{H(\pi)}{1 - \pi},
\]
so that, rearranging the terms, we have
\[
\lim_{\pi \to 1} \frac{H(\pi)}{1 - \pi} = \lim_{\pi \to 1} x_h'(\bar{u}_1(\pi)) \frac{d\bar{u}_1}{d\pi}.
\]

We now prove that \( \lim_{\pi \to 1} x_h'(\bar{u}_1(\pi)) \frac{d\bar{u}_1}{d\pi} < 0 \). Using the fact that
\[
x_h(u_1) = \frac{1}{\mu_h(v_h - c_1)} \left\{ (1 - \mu_h)(v_l - c_1)1^{-\phi} (v_l - u_1)^\phi - (1 - \mu_h)v_l + u_1 - \mu_h c_1 \right\},
\]
we have
\[
x_h'(u_1) = \frac{1}{\mu_h(v_h - c_1)} (1 - \phi(1 - \mu_h)(v_l - c_1)1^{-\phi}(v_l - u_1)^{\phi-1}).
\]
Next, since \( \bar{u}_1(\pi) \) satisfies
\[
1 = (1 - \pi) \left( \frac{v_l - c_1}{v_l - \bar{u}_1(\pi)} \right)^\phi,
\]
we have
\[
v_l - \bar{u}_1(\pi) = (1 - \pi)\frac{1}{\phi}(v_l - c_1),
\]
which implies
\[
\frac{d\bar{u}_1(\pi)}{d\pi} = \frac{1}{\phi}(1 - \pi)\frac{1}{\phi-1}(v_l - c_1)
\]
and
\[
x_h'(\bar{u}_1(\pi)) = \frac{1}{\mu_h(v_h - c_1)} \left( 1 - \phi(1 - \mu_h)(1 - \pi)^{\phi-1} \right).
\]
Combining these results, we find
\[
x_h'(\bar{u}_1(\pi)) \frac{d\bar{u}_1}{d\pi} = \frac{v_l - c_1}{\mu_h(v_h - c_1)} \left[ \frac{(1 - \pi)^{\frac{1}{\phi-1}}}{{\phi}} - (1 - \mu_h) \right]
\]
and hence
\[
\lim_{\pi \to 1} x_h'(\bar{u}_1(\pi)) \frac{d\bar{u}_1}{d\pi} = -\frac{(1 - \mu_h)(v_l - c_1)}{\mu_h(v_h - c_1)} < 0.
\]

\begin{flushright}
\text{\textbf{■}}
\end{flushright}

\subsection{A.3.2 Proof of Proposition 6}

\textbf{Proof.} We start with the form of \( W(p, \mu_h) \) given by (79) and express this instead as a function of \( \pi \). Tedious, but straightforward calculations can be used to show
\[
W(\pi, \mu_h) = (1 - \mu_h)(v_l - c_1) \left[ 1 + \frac{(v_h - c_h)2(1 - \pi)}{(v_h - c_1)(2 - \pi)} \right] + \mu_h \left[ c_h + \frac{(v_h - c_h)}{(v_h - c_1)}(v_l - c_1) \right]
\]
\[
+ \frac{(v_h - c_h)2(1 - \pi)^2}{(v_h - c_1)\pi(2 - \pi)}(v_l - c_1)\frac{\phi(\mu_h)}{1 - 2\phi(\mu_h)}((1 - \pi)^{\frac{1-2\phi(\mu_h)}{\phi(\mu_h)}} - 1).
\]
Then \( W_{\mu_h}(\pi, \mu_h) \) satisfies
\[
W_{\mu_h}(\pi, \mu_h) = \Theta + \frac{(v_h - c_h)(v_l - c_l)}{v_h - c_l} \frac{2(1-\pi)^2}{\pi(2-\pi)} \frac{d}{d\mu_h} \phi(\mu_h) \left[ \frac{1}{(1-\pi)\phi(\mu_h)} - 1 \right]
\]
where
\[
\Theta = c_h - (v_l - c_l) + \frac{v_h - c_h}{v_h - c_l} (v_l - c_l) \frac{\pi}{2 - \pi}.
\]
We will argue that when \( \pi \) is sufficiently small, then
\[
\lim_{\mu_h \to 0} W_{\mu_h}(\pi, \mu_h) < \lim_{\mu_h \to \mu_0} W_{\mu_h}(\pi, \mu_h)
\]
where \( \mu_0 \) is the value of \( \mu_h \) such that \( \phi(\mu_0) = 0 \). Inequality (83) implies that the \( W_{\mu_h} \) must be increasing on an interval of \( \mu_h \), that is, \( W \) must be convex on an interval of \( \mu_h \). In contrast, the above inequality is reversed when \( \pi \) is sufficiently close to 1.

Let
\[
M(\pi, \mu_h) = \frac{d}{d\mu_h} \phi(\mu_h) \left[ \frac{1}{(1-\pi)\phi(\mu_h)} - 1 \right]
\]
and
\[
G(\pi) = \lim_{\mu_h \to 0} M(\pi, \mu_h) - \lim_{\mu_h \to \mu_0} M(\pi, \mu_h).
\]
Since the term multiplying \( M(\pi, \mu_h) \) in (82) is positive, it suffices to show that \( G(\pi) < 0 \) for \( \pi \) close to 0 and \( G(\pi) > 0 \) for \( \pi \) close to 1. Note that
\[
\lim_{\mu_h \to 0} M(\pi, \mu_h) = - \left( \frac{v_h - c_h}{c_h - c_l} \right) \frac{\pi + \log(1-\pi)}{1-\pi}
\]
and
\[
\lim_{\mu_h \to \mu_0} M(\pi, \mu_h) = \phi'(\mu_0) \left[ -1 - \log(1-\pi) \lim_{\phi \to 0} (1-\pi)^{\frac{\phi}{\phi} - 2} \right] = \left( \frac{v_h - c_h}{c_h - c_l} \right) \frac{1}{(1-\mu_0)^2}.
\]
As a result,
\[
G(\pi) = - \left( \frac{v_h - c_h}{c_h - c_l} \right) \frac{\pi + \log(1-\pi)}{1-\pi} - \left( \frac{v_h - c_h}{c_h - c_l} \right) \frac{1}{(1-\mu_0)^2}
\]
It follows that
\[
\lim_{\pi \to 0} G(\pi) = - \left( \frac{v_h - c_h}{c_h - c_l} \right) \frac{1}{(1-\mu_0)^2}
\]
and
\[
\lim_{\pi \to 1} G(\pi) = +\infty
\]
which completes the proof. \( \blacksquare \)

A.3.3 Proof of Proposition 7

In this section, we examine the efficiency properties of equilibrium outcomes. As in the text, we define the type of seller \( i \in [0, 1] \) by \( \theta_i \in \Theta \), where \( \Theta = \{l, h\} \times \{0, 1\} \times \{0, 1\} \). The first element of \( \theta_i \) indicates whether the seller has a high or low quality good, the second element equals 1 if the seller is matched with buyer 1 and 0 otherwise, and the third element equals 1 if the seller is matched with buyer 2 and 0 otherwise. We let \( c : \Theta \to \{c_l, c_h\} \) denote the valuation a seller of type \( \theta \in \Theta \) has for her own good,

\[\text{It is straightforward to show that } W_{\mu_h}(\pi, \mu_h) \text{ is continuous.}\]

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and \( v : \Theta \rightarrow \{v_l, v_h\} \) denote the buyer’s valuation of a good purchased from a seller with type \( \theta \). Let \( \tilde{\theta} : [0, 1] \rightarrow \Theta \) denote the mapping from sellers to their respective types with \( \Theta \) representing the set of all possible mappings, \( \tilde{\theta} \).

Each buyer’s type consists of the set of sellers with whom the buyer is matched. We represent the type of buyer \( k \in \mathcal{B} = \{1, 2\} \) as a mapping \( m^k(i) : [0, 1] \rightarrow \{0, 1\} \). Let \( m^k \) denote the mapping \( m^k(i) \) and \( \mathcal{M} \) denote the set of all possible functions \( m^k \).

We model the realization of \( \tilde{\theta} \) and \( \{m^k\}_{k \in \mathcal{B}} \) as the realization of a random variable that is drawn from a known distribution.\(^{53}\) This ensures that the beliefs of each buyer and seller about the types of other buyers and sellers conditional on knowledge of their own type give rise to well defined conditional expectations, as discussed in Uhlig (1996).

An allocation is a given by \((t^k_i, x^k_i)_{k \in \mathcal{B}, i \in [0,1]}\) where \( t^k_i \in \mathbb{R} \) is a transfer of numeraire from buyer \( k \) to seller \( i \) and \( x^k_i \in [0,1] \) is the amount of good transferred from seller \( i \) to buyer \( k \). An allocation is feasible if for all \( i \) and \( k \) such that \( m^k(i) = 0 \), the allocation satisfies \( t^k_i = x^k_i = 0 \) and for all \( i \), \( x^k_i x_i^2 = 0 \). The first constraint ensures that transfers of numeraire and goods only occur between matched buyers and sellers, while the second constraint ensures that trade is exclusive.

We consider the class of direct mechanisms given by \((t^k_i, x^k_i)_{k \in \mathcal{B}, i \in [0,1]} \) where \( t^k_i : \tilde{\Theta} \times M^2 \rightarrow \mathbb{R} \) and \( x^k_i : \tilde{\Theta} \times M^2 \rightarrow [0,1] \).\(^{54}\)

**Constrained Efficiency with Direct Mechanisms.** We begin by defining and characterizing incentive compatible direct mechanisms. A direct mechanism is incentive compatible if and only if, for all sellers \( i \),

\[
\mathbb{E} \left[ \sum_{k \in \mathcal{B}} [t^k_i(\theta_i, \theta_{-i}, m^1, m^2) + (1 - x^k_i(\theta_i, \theta_{-i}, m^1, m^2))c(\theta_i)] \right] \\
\geq \mathbb{E} \left[ \sum_{k \in \mathcal{B}} [t^k_i(\tilde{\theta}_i, \theta_{-i}, m^1, m^2) + (1 - x^k_i(\tilde{\theta}_i, \theta_{-i}, m^1, m^2))c(\theta_i)] \right] \quad \forall \tilde{\theta}_i \in \tilde{\Theta}, \quad (84)
\]

and, for each buyer \( k \in \mathcal{B},

\[
\mathbb{E} \left[ \int \left[ x^k_i(\theta, m^k, m^{-k})v(\theta_i) - t^k_i(\theta, m^k, m^{-k}) \right] d\theta_i \right] \\
\geq \mathbb{E} \left[ \int \left[ x^k_i(\tilde{\theta}, \tilde{m}^k, m^{-k})v(\theta_i) - t^k_i(\tilde{\theta}, \tilde{m}^k, m^{-k}) \right] d\theta_i \right] \quad \forall \tilde{m}^k \in \tilde{\mathcal{M}} \quad (85)
\]

where the conditional expectations in (84) and (85) are taken with respect to other agents’ types.

Lastly, a direct mechanism satisfies individual rationality if and only if for all sellers \( i \),

\[
\mathbb{E} \left[ \sum_{k \in \mathcal{B}} [t^k_i(\theta_i, \theta_{-i}, m^1, m^2) + (1 - x^k_i(\theta_i, \theta_{-i}, m^1, m^2))c(\theta_i)] \right] \geq V^s(\theta_i) \quad (86)
\]

where \( V^s(\theta_i) \) denotes the expected value a seller expects to receive in equilibrium, or,

\[
V^s(\theta_i) = \begin{cases} 
\int \left[ t_{\theta_i}(u_i) + c(\theta_i)(1 - x_{\theta_i}(u_i)) \right] dF_1(u_i) & \text{if } m^1(i)m^2(i) = 0 \\
\int \left[ t_{\theta_i}(u_i) + c(\theta_i)(1 - x_{\theta_i}(u_i)) \right] d(F_1(u_i)^2) & \text{if } m^1(i)m^2(i) = 1
\end{cases}
\]

\(^{53}\)A complete description of one way to model this aggregate shock and the resulting expectations is available upon request.

\(^{54}\)The Revelation Principle applies immediately to this environment so that we may restrict attention to direct mechanisms without loss of generality.
and for each buyer $k \in B$, 
\[
E \left[ \int [x_i^k(\theta, m^k, m^{-k})v(\theta_i) - t_i^k(\theta, m^k, m^{-k})] \, di \right] \geq V^b
\]  
(87)
where $V^b$ represents the buyer’s expected equilibrium value, or 
\[
V^b = \frac{1}{2 - \pi} \sum_{i = h, l} \mu_i \left\{ (1 - \pi) \int [v_i x_i(u_l) - t_i(u_l)] \, dF_t(u_l) + \frac{\pi}{2} \int [v_i x_i(u_l) - t_i(u_l)] \, d(F_t(u_l)^2) \right\}.
\]

**Characterization.** We proceed by characterizing the set of mechanisms which maximize the sum of buyers’ utilities. First, we simplify the set of incentive constraints. Note that each seller’s match-type—i.e., whether they are matched with buyer 1, buyer 2, or both—is correlated with the buyers’ match types. As a result, it is straightforward to design a direct mechanism in which sellers have no incentives to lie about their match type, and buyers’ incentive constraints are slack. This allows us to re-write mechanisms simply as transfers (of the numerator and the good) for each of the four types of sellers: those with high or low quality goods and those matched with one or two buyers.

Imposing symmetry, we re-define the mechanism as $\{t(i, n), x(i, n)\}$ for $i = h, l$ and $n = 1, 2$ as the expected transfer and trade by a seller with quality $i$ and $n$ offers. Interim incentive compatibility requires, for each $(i, n)$
\[
t(i, n) + (1 - x(i, n))c_i \geq t(i, n) + (1 - x(i, n))c_i.
\]  
(88)

**Individual rationality of the sellers requires**
\[
t(i, 1) + (1 - x(i, 1))c_i \geq V^s(i, 1) = \int [t_i(u_l) + (1 - x_i(u_l))c_i] \, dF_t(u_l),
\]  
(89)
\[
t(i, 2) + (1 - x(i, 2))c_i \geq V^s(i, 2) = \int [t_i(u_l) + (1 - x_i(u_l))c_i] \, d(F_t(u_l)^2).
\]  
(90)

Buyers’ utility associated with any such mechanism satisfies
\[
\frac{2}{2 - \pi} \sum_{i = h, l} \mu_i \left[ (1 - \pi)(v_i x(i, 1) - t(i, 1)) + \frac{\pi}{2} (v_i x(i, 2) - t(i, 2)) \right].
\]  
(91)

Thus, a constrained efficient allocation is a feasible allocation which maximizes (91) subject to (88)–(90). It is immediate that such an allocation satisfies $x(l, n) = 1$ for $n = 1, 2$ and 
\[
t(l, n) = t(h, n) + (1 - x(h, n))c_l.
\]

That is, constrained efficient allocations do not distort trade for low-quality sellers (matched with either one or two buyers) and the incentive constraint for low-quality sellers must bind. Moreover, the individual rationality constraints for high quality sellers necessarily bind. If these constraints did not bind, one could decrease the surplus allocated to sellers of high quality goods by increasing $x(h, n)$ by $\epsilon$ and $t(h, n)$ by $\epsilon c_1$. Such a perturbation raises aggregate buyers’ payoffs by $\epsilon (v_h - c_1)$, preserves incentives, and for $\epsilon$ small does not violate individual rationality of high quality sellers.

**Proof of Proposition 7.** We first prove that equilibrium allocation is constrained efficient when $\phi_1 > 0$. To start, note that the individual rationality constraint for low-quality sellers must bind, otherwise one can improve buyers’ payoffs by reducing transfers to low-quality sellers and adjusting trade with high quality sellers to preserve incentive compatibility. Since $\phi_1 > 0$, such a perturbation raises buyers’ utility. Summarizing the results above, when $\phi_1 > 0$ the solution to the program described above must satisfy
\[ x(l, n) = 1, \]
\[ t(l, n) = V^s(l, n), \quad (92) \]
\[ t(h, n) + (1 - x(h, n))c_h = V^s(h, n), \quad \text{and} \quad (93) \]
\[ t(l, n) = t(h, n) + (1 - x(h, n))c_l. \quad (94) \]

We now show that expected volume of trade in a constrained efficient allocation coincides with expected trade in our equilibrium. It is clear that trade by low-quality sellers is the same (since \( x(l, n) = 1 \) for \( n = 1, 2 \)). To see that trade by high quality sellers also coincides, first note that (92)–(94) imply
\[ V^s(h, n) - V^s(l, n) = [1 - x(h, n)](c_h - c_l). \quad (95) \]

Using the definition of \( V(i, n) \) in (89)–(90), along with the fact that each menu in equilibrium satisfies the low-quality seller’s incentive constraint with equality, we have
\[ V^s(h, n) - V^s(l, n) = \int [1 - x_h(u_1)](c_h - c_l) d(F_l(u_1)^n). \quad (96) \]

Solving (95)–(96), we see that the volume of trade under the optimal mechanism between buyers and high quality sellers with \( n \) offers is
\[ x(h, n) = \int x_h(u_1) d(F_l(u_1)^n). \quad (97) \]

Using (97), similar algebra reveals that the transfers satisfy
\[ t(i, n) = \int t_i(u_1) d(F_l(u_1)^n). \quad (98) \]

An immediate consequence of (97) and (98) is that buyers’ utility coincides with what they receive in equilibrium, which proves the claim for \( \phi_l > 0 \).

Consider next the case of \( \phi_l < 0 \). We claim that equilibrium is constrained efficient if, and only if, \( x_h(u_1) = 1 \) for all \( u_1 \in \text{Supp}(F_l) \). To see why, suppose that the equilibrium satisfies
\[ \int x_h(u_1) d(F_l(u_1)^n) < 1 \]
for some \( n \in \{1, 2\} \). We will show that a perturbation of such an allocation is feasible and increases buyers’ utility, i.e. the initial allocation cannot be constrained efficient. To do so, consider the mechanism
\[ t(l, n) = V^s(l, n) \]
\[ x(l, n) = 1 \]
\[ t(h, n) = \int t_h(u_1) d(F_l(u_1)^n) \]
\[ x(h, n) = \int x_h(u_1) d(F_l(u_1)^n). \]

This mechanism satisfies the incentive and individual rationality constraints by construction. Now con-
sider the following perturbation: for some $n$ and $\epsilon > 0$, let

$$\hat{t}(l, n) = t(l, n) + (c_h - c_l)\epsilon$$

$$\hat{x}(l, n) = 1$$

$$\hat{t}(h, n) = t(h, n) + c_h\epsilon$$

$$\hat{x}(h, n) = x(h, n) + \epsilon.$$  

We argue that this perturbation remains feasible and strictly increases buyers’ utilities. Note that incentive constraints are satisfied since

$$\hat{t}(l, n) = t(l, n) + (c_h - c_l)\epsilon \geq t(h, n) + (1 - x(h, n))c_l + (c_h - c_l)\epsilon = \hat{t}(h, n) + (1 - \hat{x}(h, n))c_l.$$  

Moreover, this perturbation raises the payoff of low-quality sellers and leaves the payoff of high quality sellers unchanged. Finally, buyers’ payoffs rise since the net impact of this perturbation is given by

$$[\mu_h(v_h - c_h) - \mu_l(v_h - c_l)] \epsilon = -\phi_1 \mu_l(c_h - c_l) > 0,$$

where the last inequality follows from $\phi_1 < 0$.

The final step of the proof requires showing that a pooling equilibrium—with $x_h(u_1) = 1$ for all $u_1 \in \text{Supp}(F_1)$—is constrained efficient. To see why, note that $V^*(l, n) = V^*(h, n)$ for $n \in \{1, 2\}$ in any incentive compatible mechanism with full trade. Since the sellers’ participation constraint binds, and the total surplus generated by the constrained efficient allocation coincides with that in the equilibrium, the payoff to the buyers must coincide as well.

### A.4 Proofs from Section 6

#### A.4.1 Proof of Proposition 8

Given $\hat{\pi}^1 = \hat{\pi}^2 = \hat{\pi}$, we can use the analysis of our benchmark model to characterize the unique equilibrium. In particular, substituting $\hat{\pi} = \pi$, Propositions 2 and 3 characterize the equilibrium offer distributions, $\{F_j|u_j\}$, along with equilibrium profits, which we denote by $\Pi^*(\pi)$. For any $\phi_1 < 1$, equilibrium profits are continuous and strictly decreasing in $\pi$, with $\Pi^*(0) > 0$ and $\Pi^*(1) = 0$. By assumption, $C'(\pi)$ is a continuous, strictly increasing function with $C'(0) = 0$ and $C'(1) > 0$, so that there is a unique solution to the first order condition $C'(\pi) = \Pi^*(\pi)$.

#### A.4.2 Proof of Lemma 5

Consider first the case of $\phi_1 \geq 0$. In this region of the parameter space, $\pi^*$ satisfies

$$C'(\pi^*) - (1 - \mu_h)(1 - \pi^*)(v_l - c_l) = 0,$$

and hence clearly $\frac{d\pi^*}{d\mu_h} < 0$.

Next consider the case of $\phi_1 < 0$. If $\phi_1 \leq \phi_1$, where $\phi_1 < 0$ is defined in (55), then $\pi^*$ satisfies

$$C'(\pi^*) - (1 - \pi^*)(\mu_h v_h + \mu_l v_l - c_h) = 0,$$

and hence clearly $\frac{d\pi^*}{d\mu_h} > 0$. The more difficult case is when $\phi_1 < \phi_1 < 0$. In this case, $\pi^*$ satisfies

$$C'(\pi^*) - (1 - \pi^*) [\mu_l(v_l - \hat{u}_l) + \mu_h \Pi_h(\hat{u}_l, c_h)] = 0,$$

where $\hat{u}_l$ satisfies $\gamma(\hat{u}_l, \mu_h) = 0$, with

$$\gamma(\hat{u}_l, \mu_h) = \bar{v} - [v_l + (1 - \pi)g(\mu_h, \pi)(\hat{u}_l - v_l)] - (1 - \pi) [\mu_l(v_l - \hat{u}_l) + \mu_h \Pi_h(\hat{u}_l, c_h)].$$
and
\[ g(\mu_h, \pi) = (1 - \pi)^{\frac{1}{\Phi_l}}. \]

To prove that \( \frac{d\pi}{d\mu_h} > 0 \), we will show that
\[ \Pi^*(\pi, \mu_h) = (1 - \pi) [\mu_l (v_l - \hat{u}_l) + \mu_h \Pi_h (\hat{u}_l, c_h)] \]
is increasing in \( \mu_h \) and decreasing in \( \pi \). To prove the first result, note that
\[ \frac{\partial \gamma}{\partial \hat{u}_l} = - \frac{(1 - \pi)^{\frac{1}{\Phi_l}}}{\mu_l} + (1 - \pi) \phi_l < 0, \]

since we are looking at the region with \( \phi_l < 0 \), and
\[
\begin{align*}
\frac{\partial \gamma}{\partial \mu_h} &= (v_h - v_l) - (1 - \pi) \left[ (\hat{u}_l - v_l) \left( \frac{\partial g}{\partial \mu_h} + 1 \right) + \Pi_h (\hat{u}_l, c_h) \right] \\
&\geq (v_h - v_l) - (1 - \pi) [(\hat{u}_l - v_l) + \Pi_h (\hat{u}_l, c_h)] \\
&= \pi (v_h - v_l) + (1 - \pi) (c_h - \hat{u}_l) \left( \frac{v_h - c_l}{c_h - c_l} \right) \geq 0,
\end{align*}
\]

where we use that \( \frac{\partial g}{\partial \mu_h} < 0 \) in the first inequality and \( c_h \geq \hat{u}_l \) in the last. Hence, \( \frac{d\hat{u}_l}{d\mu_h} \geq 0 \) and
\[
\begin{align*}
\frac{d\Pi^*}{d\mu_h} &= (1 - \pi) \left[ \Pi_h (\hat{u}_l, c_h) + (\hat{u}_l - v_l) - \mu_l \frac{d\hat{u}_l}{d\mu_h} \phi_l \right] \geq 0.
\end{align*}
\]

To show that \( \Pi^* \) is decreasing in \( \pi \), we must show that
\[
\frac{1}{\phi_l} (1 - \pi)^{\frac{1}{\Phi_l}} (\pi - v_l) - (1 - \pi)^{\frac{1}{\Phi_l}} \frac{d\hat{u}_l}{d\pi} \leq 0,
\]
or
\[
\frac{d\hat{u}_l}{d\pi} \geq \frac{\hat{u}_l - v_l}{\phi_l (1 - \pi)}.
\]

Solving (102) explicitly for \( \hat{u}_l \) yields
\[
\begin{align*}
\hat{u}_l &= \frac{(v_h - v_l) + (1 - \pi)^{\frac{1}{\Phi_l}} (\pi - v_l) \left( v - \mu_h \frac{v_h - c_l}{c_h - c_l} \right)}{(1 - \pi)^{\frac{1}{\Phi_l}} - \mu_l \phi_l (1 - \pi)} \\
&= v_l + \frac{\mu_h (v_h - v_l) - (1 - \pi) \mu_h (v_l - c_l) (v_h - c_h)}{(1 - \pi)^{\frac{1}{\Phi_l}} - \mu_l \phi_l (1 - \pi)} \mu_h \frac{v_l - c_l}{c_h - c_l},
\end{align*}
\]

so that
\[
\begin{align*}
\frac{d\hat{u}_l}{d\pi} &= \frac{\mu_h (v_l - c_l) (v_h - c_h)}{(1 - \pi)^{\frac{1}{\Phi_l}} - \mu_l \phi_l (1 - \pi)} - (\hat{u}_l - v_l) \frac{1}{\phi_l} (1 - \pi)^{\frac{1}{\Phi_l}} + \mu_l \phi_l.
\end{align*}
\]
Thus we have to show that
\[
\frac{\mu_h (v_l - c_l)(v_h - c_h)}{c_h - c_l} - \frac{1}{\phi_l} \frac{(\hat{u}_l - v_l)}{(1 - \pi) \frac{\phi_l}{\phi_l} - \mu_l (1 - \pi)} \phi_l (1 - \pi) \geq \frac{\hat{u}_l - v_l}{(1 - \pi) \frac{\phi_l}{\phi_l} - \mu_l (1 - \pi)}.
\]

Again, since \(\hat{u}_l \geq v_l\) and \(\phi_l \leq 0\), the right hand side of the inequality above is negative and the left hand side is positive. This completes the proof.

A.4.3 Proof of Lemma 6

Let
\[
X(\pi) = \left[ 2\pi (1 - \pi) + \pi^2 \right] \left( \mu_h X_h(\pi)(v_h - c_h) + \mu_l (v_l - c_l) \right)
\]
denote the gains from trade realized in a symmetric equilibrium with \(\pi^* = \pi\). Welfare is defined as
\[
W(\tau) = -2Ac(\pi(\tau)) + X(\pi(\tau)),
\]
so that
\[
W'(\tau) = \left[ X'(\pi) - 2Ac'(\pi) \right] \frac{d\pi}{d\tau}.
\]

Since
\[
\lim_{\lambda \to 0} \pi^* = 1,
\]
and
\[
\lim_{\pi \to 1} X(\pi) < 0,
\]
there exists \(\bar{\lambda} > 0\) such that \(X'(\pi) < 0\) whenever \(\lambda < \bar{\lambda}\). Since \(\frac{d\pi}{d\tau} < 0\), it follows immediately from (106) that \(W'(\tau) > 0\) in this region. ■

A.5 Proofs from Section 7

A.5.1 Poisson Meeting Technology

It follows immediately from (31) that
\[
Q_n(\alpha) = \frac{e^{-\alpha} \alpha^{n-1}}{(n-1)!}
\]
and \(Q_0(\alpha) = 1 - \sum_{n=1}^{\infty} Q_n(\alpha) = 0\). From (34), \(\tilde{\pi} = 1 - Q_1(\alpha)\) and substituting into (35) implies
\[
G_1(u_l; \alpha) = \frac{1}{1 - Q_1(\alpha)} \sum_{n=2}^{\infty} Q_n(\alpha) F_l^{n-1}(u_l; \alpha)
\]
\[
= \frac{Q_1(\alpha)}{1 - Q_1(\alpha)} \left[ e^{\alpha F_l(u_l; \alpha)} - 1 \right].
\]

Next, using the solution to the differential equation (36),
\[
1 - \tilde{\pi} + \tilde{\pi} G_1(u_l; \alpha) = (1 - \tilde{\pi}) \left( \frac{v_l - u_l}{v_l - c_l} \right)^{-\phi_l}.
\]
one can show that
\[ G_1(u_1; \alpha) = \frac{Q_1(\alpha)}{1 - Q_1(\alpha)} \left[ \left( \frac{v_1 - u_1}{v_1 - c_1} \right)^{-\phi_1} - 1 \right]. \]  

Combining (107) and (108), we obtain
\[ F_1(u_1; \alpha) = \frac{-\phi_1}{\alpha} \log \left( \frac{v_1 - u_1}{v_1 - c_1} \right). \]

Note also that \( F_1(\hat{u}_1; \alpha) = 1 \) implies
\[ \frac{v_1 - \hat{u}_1}{v_1 - c_1} = e^{-\frac{\alpha}{\phi_1}}, \]  
and
\[ f_1(u_1; \alpha) = \frac{\phi_1}{\alpha} (v_1 - u_1)^{-1}, \]
which we use below.

Next, we evaluate the utilitarian welfare measure given the Poisson meeting technology:
\[
\sum_{n=1}^{\infty} \frac{P_n(\alpha)}{n} \left[ \mu_h(v_h - c_h) \int x_n(u_1) \, d(F_1^n(u_1; \alpha)) + \mu_t(v_t - c_1) \right] \\
= \mu_h(v_h - c_h) \int x_n(u_1) \sum_{n=1}^{\infty} nP_n(\alpha)F_1^{n-1}(u_1; \alpha)f_1(u_1; \alpha) \, du_1 + \mu_t(v_t - c_1) \sum_{n=1}^{\infty} P_n(\alpha) \\
= \mu_h(v_h - c_h) \int x_n(u_1) \sum_{n=1}^{\infty} nP_n(\alpha)F_1^{n-1}(u_1; \alpha)f_1(u_1; \alpha) \, du_1 + \mu_t(v_t - c_1)(1 - e^{-\alpha}).
\]

Consider
\[ \hat{W}(\alpha) = \int x_n(u_1) \sum_{n=1}^{\infty} nP_n(\alpha)F_1^{n-1}(u_1; \alpha)f_1(u_1; \alpha) \, du_1. \]  

Substituting for \( nP_n(\alpha) \), and using (35) and (108), we obtain
\[
\hat{W}(\alpha) = \int x_n(u_1) \alpha \sum_{n=1}^{\infty} Q_n(\alpha)F_1^{n-1}(u_1; \alpha)f_1(u_1; \alpha) \, du_1 \\
= \int x_n(u_1) \alpha \left[ Q_1(\alpha) + (1 - Q_1(\alpha))G_1(u_1; \alpha) \right] f_1(u_1; \alpha) \, du_1 \\
= \int x_n(u_1) \alpha Q_1(\alpha) \left( \frac{v_1 - u_1}{v_1 - c_1} \right)^{-\phi_1} f_1(u_1; \alpha) \, du_1.
\]

Substituting for \( f_1(u_1) \) and \( x_n(u_1) \) and re-arranging terms yields
\[ \hat{W}(\alpha) = Q_1(\alpha) \frac{\phi_1(v_1 - c_1)^{\phi_1}}{\mu_h(v_h - c_h)} \int \left( v_1 - u_1 \right)^{-1 - \phi_1} \left[ \mu_t(v_t - c_1)^{1 - \phi_1} \left( v_1 - u_1 \right)^{\phi_1} - (v_1 - u_1) + \mu_h(v_t - c_1) \right] \, du_1 \]
where the limits of integration are \( c_1 \) and \( \hat{u}_1(\alpha) \). Applying tedious but straightforward calculus to compute the integral yields
\[ \hat{W}(\alpha) = e^{-\alpha} \frac{\phi_1}{\mu_h(v_h - c_h)} \left[ \mu_t(v_t - c_1)^{\frac{\alpha}{\phi_1}} + \frac{v_1 - c_1}{1 - \phi_1} \left( e^{-\alpha^{1-\phi_1}} - 1 \right) \right] + \frac{v_1 - c_1}{v_h - c_1} - e^{-\alpha} \frac{v_1 - c_1}{v_h - c_1}. \]
Evaluating welfare as a function of $\alpha$ then implies
\[
W(\alpha) = \mu_h(v_h - c_h) \left( \frac{v_l - c_l}{v_h - c_l} + \frac{\mu_h(v_h - c_l)}{\mu_h(v_l - c_l)} \right) e^{-\alpha} + \frac{\phi_l}{1 - \phi_l} \frac{v_l - c_l}{\mu_h(v_l - c_l)} \left( e^{-\frac{v_l}{\phi_l} - e^{-\alpha}} - e^{-\alpha} \frac{v_l - c_l}{v_h - c_l} \right) + (1 - e^{-\alpha}) \mu_l(v_l - c_l)
\]
Differentiating welfare with respect to $\alpha$ and re-arranging terms, we obtain
\[
W'(\alpha) = e^{-\alpha} \frac{(v_h - c_h)(v_l - c_l)}{v_h - c_l} \left[ \mu_l(1 - \alpha) - \frac{1}{(1 - \phi_l)} e^{-\alpha} \frac{\phi_l}{\phi_l} + \frac{\phi_l}{1 - \phi_l} + \mu_h + \mu_l \frac{v_h - c_h}{v_l - c_l} \right]
\]
Let $H(\alpha)$ denote the term in brackets in the equation above. Since $H(\alpha)$ is a strictly concave function with $H(0) > 0$ and $\lim_{\alpha \to \infty} H(\alpha) = -\infty$, there exists a unique $\alpha^*$ such that for all $\alpha > \alpha^*$, $H(\alpha) < 0$. Hence, for all finite $\alpha > \alpha^*$, $W'(\alpha) < 0$.

A.5.2 Geometric Meeting Technology

For the Geometric meeting technology with $\lambda(\alpha) = \alpha/(1 - \alpha)$, we have
\[
Q_n(\alpha) = (1 - \alpha)^2 n \alpha^{n-1}
\]
and $Q_0(\alpha) = 0$. Much as in the Poisson case, one can use (35) and (36) to show
\[
F_1(u_1; \alpha) = \frac{1}{\alpha} \left[ 1 - \left( \frac{v_l - u_1}{v_l - c_l} \right)^{\phi_l} \right],
\]
so that
\[
f_1(u_1; \alpha) = \frac{\phi_l}{2 \alpha} \left( \frac{v_l - u_1}{v_l - c_l} \right)^{\phi_l} - \frac{1}{v_l - u_1}
\]
and the upper bound $\bar{u}_1(\alpha)$ satisfies
\[
\left( \frac{v_l - \bar{u}_1(\alpha)}{v_l - c_l} \right)^{\phi_l} = 1 - \alpha.
\]

Next, we evaluate the utilitarian welfare measure given the Geometric meeting technology:
\[
\sum_{n=1}^{\infty} p_n(\alpha) \left[ \mu_h(v_h - c_h) \int x_h(u_1) d(F_1^n(u_1; \alpha)) + \mu_l(v_l - c_l) \right]
\]
\[
= \mu_h(v_h - c_h) \int x_h(u_1) \sum_{n=1}^{\infty} n p_n(\alpha) F_1^{n-1}(u_1; \alpha) f_1(u_1; \alpha) du_1 + \mu_l(v_l - c_l) \alpha.
\]

Consider $\hat{W}(\alpha)$, as defined in (110). Using similar steps to those we used above, one can show that
\[
\hat{W}(\alpha) = (1 - \alpha) \frac{\phi_l(v_l - c_l) \phi_l}{2 \mu_h(v_l - c_l)} \int_{c_l}^{\bar{u}_1(\alpha)} (v_l - u_1)^{-1 - \phi_l} \left[ \mu_l(v_l - c_l)^{1 - \phi_l} (v_l - u_1)^{\phi_l} - (v_l - u_1) + \mu_h(v_l - c_l) \right] du_1
\]
\[
= \frac{\phi_l}{2 \mu_h(v_l - c_l)} (v_l - c_l) \left\{ \frac{\mu_l}{\phi_l} \frac{2}{\phi_l} + \frac{1}{1 - \phi_l/2} \left[ (1 - \alpha)^2 / \phi_l^{-1} - 1 \right] + \mu_h \frac{\alpha}{1 - \alpha \phi_l} \right\}.
\]
Therefore, welfare as a function of $\alpha$ is given by
\[
W(\alpha) = \frac{\phi_1 (1 - \alpha) (v_h - c_h)(v_l - c_l)}{2 (v_h - c_l)} \left\{ \mu_1 \frac{\alpha^2}{\phi_1} + \frac{1}{1 - \phi_1/2} \left[ (1 - \alpha)^2/\phi_1 - 1 \right] + \mu_h \frac{\alpha^2}{1 - \alpha} \right\} + \alpha \mu_1 (v_l - c_l)
\]

Differentiating with respect to $\alpha$ and re-arranging terms, we obtain
\[
W' (\alpha) = \mu_1 (v_l - c_l) + \frac{(v_h - c_h)(v_l - c_l)}{(v_h - c_l)} \left[ \mu_1 (1 - 2\alpha) - \frac{1}{1 - \phi_1/2} (1 - \alpha)^\phi_1 - 1 + \frac{\phi_1/2}{1 - \phi_1/2} + \mu_h \right].
\]

Note that $W'(0) = \mu_1 (v_l - c_l)$ and
\[
W'(1) = \mu_1 (v_l - c_l) \frac{c_h - c_l}{v_h - c_l} + \frac{(v_h - c_h)(v_l - c_l)}{(v_h - c_l)} \left[ \mu_h + \frac{\phi_1/2}{1 - \phi_1/2} \right].
\]

Since $W'(0) > 0$ and $W'(1) > 0$ and $W'(\alpha)$ is a strictly concave function of $\alpha$, there exists no $\alpha \in (0, 1)$ such that $W'(\alpha) < 0$.

### A.6 Proofs from Section 8

#### A.6.1 Construction of Equilibrium for the Insurance Model

The construction of equilibrium follows the logic of Section 4. For brevity, we focus on the region of the parameter space where all equilibrium menus are separating and involve no cross-subsidization. This obtains when the fraction of type-b agents, $\mu_b$, is sufficiently large. The optimality conditions with respect to $u_b$ and $u_g$ in this case are

\[
\frac{\pi f_b(u_b)}{1 - \pi + \pi F_b(u_b)} \Pi_b (u_b) = C'(u_b) - \frac{\mu_g}{\mu_b} \left[ \frac{\theta_g (1 - \theta_g)}{\theta_b - \theta_g} C'(u^g_b) - \frac{\theta_g (1 - \theta_g)}{\theta_b - \theta_g} C'(u^a_g) \right]
\]

\[
\frac{\pi f_g(u_g)}{1 - \pi + \pi F_g(u_g)} \Pi_g (u_b, u_g) = \frac{(1 - \theta_g) \theta_b / \theta_b - \theta_g}{\theta_b - \theta_g} C'(u^g_g) - \frac{\theta_g (1 - \theta_b)}{\theta_b - \theta_g} C'(u^a_g).
\]

These two differential equations, along with the boundary conditions $F_j(u_j) = 0$ with $u_j = \theta_j w(y - d) + (1 - \theta_j) w(y)$, characterize the equilibrium. Note that these are similar in structure to (21), except that the marginal cost of delivering utility varies with the level of utility (this was constant in the linear model). To solve this system, we make use of the SRP relationship, $F_b(u_b) = F_g(U_g(u_b))$, which implies $f_b(u_b) = f_g(U_g(u_b))U'_g(u_b)$. Dividing the first differential equation by the second and using the SRP identities, we obtain

\[
\frac{\Pi_b (u_b) U'_g(u_b)}{\Pi_g (u_b, U_g(u_b))} = \frac{C'(u_b) - \frac{\mu_g}{\mu_b} \left[ \frac{\theta_g (1 - \theta_g)}{\theta_b - \theta_g} C'(u^g_b) - \frac{\theta_g (1 - \theta_g)}{\theta_b - \theta_g} C'(u^a_g) \right]}{\frac{(1 - \theta_g) \theta_b / \theta_b - \theta_g}{\theta_b - \theta_g} C'(u^g_g) - \frac{\theta_g (1 - \theta_b)}{\theta_b - \theta_g} C'(u^a_g)},
\]

where $u^a_g$ and $u^a_g$ are related to $u_b$ and $U_g$ through (37). Equation (113) is thus an ordinary differential equation in $U_g$, along with the boundary condition $U_g(U_b) = u_g$. Note that this does not depend on $\pi$. Given $U_g$, equations (111) – (112) can be solved for the distribution functions.

Given a functional form for the utility function, $w$, this system can be solved numerically. Figure 9 depicts the solution for the following parameterization: $w(c) = \sqrt{2c}$, $y = 10$, $d = 9$, $\theta_b = 0.9$, $\theta_g = 0.6$, $\mu_g = 0.3$. The left panel plots the equilibrium $U_g$, while the right panel shows the resource losses associated with imperfect insurance—specifically, the function $L(u_b)$ from (38).
A.6.2 Type-Specific \( \pi \)

Since our proofs that \( F_h \) and \( F_1 \) have no flat regions and \( F_h \) has no mass points immediately extend to the case when \( \pi_l \neq \pi_h \), we omit them in the interest of brevity. Hence, we begin by analyzing the potential for mass point equilibria; that is, for \( F_1(\cdot) \) to feature a mass point—to emerge when \( \pi_l \neq \pi_h \).

**Proposition 10.** Suppose \( \pi_l < \pi_h \). Then \( F_1(\cdot) \) does not have a mass point.

**Proof.** We prove a profitable deviation exists much as in the case when \( \pi_l = \pi_h \). In particular, in any such equilibrium with a mass point, \( \Pi_l = 0 \) and the following inequalities must hold

\[
-\mu_h \left( 1 - \pi_h + \pi_h F_l^- (\hat{u}_l) \right) \frac{v_h - c_h}{c_h - c_l} + \mu_l \left( 1 - \pi_l + \pi_l F_l^- (\hat{u}_l) \right) \leq 0
\]

\[
\mu_h \left( 1 - \pi_h + \pi_h F_l^+ (\hat{u}_l) \right) \frac{v_h - c_h}{c_h - c_l} - \mu_l \left( 1 - \pi_l + \pi_l F_l^+ (\hat{u}_l) \right) \leq 0.
\]

Rearranging the above, we must have

\[
\frac{1 - \pi_l + \pi_l F_l^- (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^- (\hat{u}_l)} \leq \frac{\mu_h}{\mu_l} \frac{v_h - c_h}{c_h - c_l} \leq \frac{1 - \pi_l + \pi_l F_l^+ (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^+ (\hat{u}_l)}. \tag{114}
\]

Since \( F_l^+ (\hat{u}_l) \geq F_l^- (\hat{u}_l) \) and \( \pi_l < \pi_h \), then we must have that

\[
\frac{1 - \pi_l + \pi_l F_l^- (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^- (\hat{u}_l)} > \frac{1 - \pi_l + \pi_l F_l^+ (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^+ (\hat{u}_l)}
\]

which is a contradiction. \( \blacksquare \)

**Proposition 11.** Suppose \( \pi_l > \pi_h \). If a mass points exists, then \( F_1(v_l) = 1 \).

**Proof.** First, it is immediate that a mass point cannot exist for any \( u_l \neq v_l \). Hence, suppose by way of contradiction that there is a mass on \( v_l \) that is not full. Then either \( F_l^- (v_l) > 0 \) or \( F_l^+ (v_l) < 1 \). Since above and below \( v_l \), the equilibrium features no mass points, the equilibrium must also satisfy the strict rank-preserving property. Let \( S = \{(v_l, u_h)\} \) and note that \( S \) must have positive measure. Furthermore, the set \( S \) must be of the form \( \{ (v_l, u_h) : u_h \in [\hat{u}_h, \bar{u}_h] \} \). Note that we have, \( \hat{u}_h > \bar{u}_h \geq c_h > v_l \).

Therefore, in a neighborhood around \( S \), all equilibrium menus should be separating. As a result, they must satisfy the optimality condition with respect to \( u_l \)—for values of \( u_l \in [v_l - \varepsilon, v_l + \varepsilon] \setminus \{v_l\} \) for
small but positive $\varepsilon$ (depending on whether mass is above or below $v_l$):

$$-\mu_l (1 - \pi_l + \pi_l F_l (u_l)) + \mu_l \pi_l f_l (u_l) (v_l - u_l) + \mu_h (1 - \pi_h + \pi_h F_h (u_h)) \frac{v_h - c_h}{c_h - c_l} = 0.$$  

Using SRP,

$$-\mu_l (1 - \pi_l + \pi_l F_l (u_l)) + \mu_l \pi_l f_l (u_l) (v_l - u_l) + \mu_h (1 - \pi_h + \pi_h F_h (u_h)) \frac{v_h - c_h}{c_h - c_l} = 0.$$  

Therefore, if positive mass is above $v_l$, we must have that

$$\mu_h (1 - \pi_h + \pi_h F_l (u_l)) \frac{v_h - c_h}{c_h - c_l} - \mu_l (1 - \pi_l + \pi_l F_l (u_l)) > 0,$$

and if it is below,

$$\mu_h (1 - \pi_h + \pi_h F_l (u_l)) \frac{v_h - c_h}{c_h - c_l} - \mu_l (1 - \pi_l + \pi_l F_l (u_l)) < 0.$$  

From above, if mass point is to be an equilibrium property, the inequality (114) must hold:

$$\frac{1 - \pi_l + \pi_l F_l^- (v_l)}{1 - \pi_h + \pi_h F_l^- (v_l)} \leq \frac{\mu_l}{\mu_1} \frac{v_l - c_l}{c_l - c_l} \leq \frac{1 - \pi_l + \pi_l F_l^+ (v_l)}{1 - \pi_h + \pi_h F_l^+ (v_l)} < \frac{\pi_l}{\pi_h}.$$  

(115)

Now suppose that $F_l^+ (v_l) < 1$. Then, from the differential equation above,

$$F_l (u_l) \left[ \mu_l \pi_l \frac{v_l - c_l}{c_l - c_l} \frac{1 - \pi_l}{\mu_1} \right] - \mu_l \pi_l f_l (u_l) (u_l - v_l) + \mu_h (1 - \pi_h) \frac{v_h - c_h}{c_h - c_l} - \mu_l (1 - \pi_l) = 0.$$  

The general solution to the above differential equation is given by

$$F_l (u_l) = A_1 (u_l - v_l) \frac{\mu_l \pi_l (v_l - c_l) - \pi_l u_l}{\mu_l \pi_l} + A_2.$$  

Since \( \frac{\mu_l \pi_l (v_l - c_l) - \pi_l u_l}{\mu_l \pi_l} < 0 \) from (115), the above expression approaches either $\pm\infty$ as $u_l$ approaches $v_l$ from above. Hence, $F_l^+ (v_l) < 1$ cannot hold.

Now suppose that $F_l^- (v_l) > 0$. Then, similar to above, we must have that

$$F_l (u_l) = A_1 (v_l - u_l) \frac{\mu_l \pi_l (v_l - c_l)}{\mu_l \pi_l} + A_2.$$  

As $u_l$ converges to $v_l$, the above expression converges to $\infty$, which is in contradiction with $F_l^- (v_l) < 1$. This proves the claim.

\[ \square \]

### A.6.3 Proof of Proposition 9

We have already shown a masspoint equilibrium, if it exists, must full mass at $v_l$. Now, the worst menu in a masspoint equilibrium (i.e., the one with the lowest $u_h$) must set $u_h = c_h$ (otherwise, lowering $u_h$ strictly raises profits). By construction, a function $F_h$ that satisfies (40) ensures equal profits at all points in the support. To rule out other deviations, consider the payoff from offering $u_h' = v_l - \varepsilon$, $u_h' \in$
$[u_l, u_h]$. The change in profits (per $\epsilon$) satisfy

$$
\mu_l (1 - \pi_l) - (1 - \pi_h + \pi_h F_h) \mu_h \frac{v_h - c_h}{c_h - c_l} = 1 - \frac{(1 - \pi_h + \pi_h F_h) \mu_h v_h - c_h}{(1 - \pi_l) \mu_l (1 - \pi_l)}.
$$

It is sufficient to show that this is negative at the bottom, i.e., when $F_h = 0$, which leads to

$$
1 - \frac{(1 - \pi_h) \mu_h v_h - c_h}{(1 - \pi_l) \mu_l (1 - \pi_l)} < 0 \Rightarrow \frac{1 - \pi_l}{1 - \pi_h} < 1 - \phi.
$$

To rule out equilibria without masspoints, note that, in such an environment, the equilibrium is strictly rank-preserving, so there must be a worst menu, i.e., one with $F = 0$. If it is a pooling menu, then it must offer $u_h = u_l = c_h$. In other words, $\Pi_l = v_l - c_h < 0$. On the other hand, if it is a separating one, it must satisfy the FOC for $u_l$:

$$
\frac{\pi_l f_l}{1 - \pi_l + \pi_l F_l} \Pi_l = 1 - \left( \frac{1 - \pi_h}{1 - \pi_l} \right) (1 - \phi_l) < 0 \Rightarrow \Pi_l < 0
$$

i.e., the worst menu in a non-masspoint equilibrium must necessarily lose money on the low type. But then, the best menu must also lose money, because

$$
\Pi_l (u_l) = v_l - u_l < v_l - u_l < 0.
$$

Now, consider a deviation of the form $(\bar{u}_l - \epsilon, \bar{u}_h)$ changes profits, relative to $(\bar{u}_l, \bar{u}_h)$, by

$$
\mu_l - \mu_h \frac{v_h - c_h}{c_h - c_l} - \mu_l f_l \Pi_l (u_l) = \mu_l \phi - f_l \Pi_l (u_l) > 0
$$

yielding the desired contradiction. Thus, in a masspoint equilibrium, the distribution of $u_l$ is degenerate at $v_l$, i.e., buyers make zero profits from type-$l$ sellers. A buyer can deviate and offer a lower $u_l$, but that brings higher profits only from the captive $l$-types at the expense of lower profits from both captive and noncaptive $h$-types. When the condition in part (1) of the proposition is satisfied, $\pi_l$ is sufficiently high or equivalently, the fraction of captive $l$-types is too low to make such a deviation attractive.

### A.6.4 Equilibrium with vertical differentiation

Here, we conjecture and characterize an equilibrium with vertical differentiation. We restrict attention to the region of the parameter space where both buyers offer separating contracts without cross-subsidization. First, note that the upper and lower bounds of the distributions of both buyers must coincide, i.e., the distributions of offers by both buyers have the same support. This then implies that $F_{ij}$ has mass of $\alpha$ at its lowest point $c_l$. To see this, consider the equal profit condition for each buyer (recall that all ties are resolved in favor of buyer 1):

$$
(1 - \pi) (v_l - c_l) = \Pi (\bar{u}_l, \bar{u}_h)
$$

$$
(1 - \pi + \pi \alpha) (v_l - c_l + B) = \Pi (\bar{u}_l, \bar{u}_h) + B.
$$

Solving, we obtain $\alpha = \frac{B}{B + v_l - c_l}$. Next, we posit that (i) $U^1_h (u_l)$ is strictly increasing everywhere in the support (ii) $U^2_h (u_l) = c_h$ for $u_l \in [c_l, c_l + s]$, $s \geq 0$. In the interval $(c_l + s, \bar{u}_l]$, $U^2_h (u_l)$ is strictly increasing. Formally, the distributions $F^k_j$ satisfy the strict rank-preserving conditions

$$
F^1_l (u_l) = F^1_h (U^1_h (u_l)) \quad u_l \in [u_l, \bar{u}_l] (116)
$$

$$
F^2_l (u_l) = F^2_h (U^2_h (u_l)) \quad u_l \in (c_l + s, \bar{u}_l]. (117)
$$
The optimality conditions for \( u_1 \) and \( u_h \) for the two buyers yield:

\[
\frac{\pi f^2_i(u_1)}{1 - \pi + \pi f^2_i(u_1)} \Pi^1_i(u_1) = 1 - \frac{\mu_h}{\mu_l} \left( 1 - \pi + \pi f^2_i(u_1) \right) \frac{v_h - c_h}{c_h - c_l} \tag{118}
\]

\[
\frac{\pi f^2_h(u_h)}{1 - \pi + \pi f^2_h(u_h)} \Pi^1_h(u_1, U^2_h(u_1)) = \frac{v_h - c_l}{c_h - c_l} \tag{119}
\]

\[
\frac{\pi f^1_i(u_1)}{1 - \pi + \pi f^1_i(u_1)} (v_1 - u_1) = 1 - \frac{\mu_h}{\mu_l} \left( 1 - \pi + \pi f^1_i(u_1) \right) \frac{v_h - c_h}{c_h - c_l} \tag{120}
\]

\[
\frac{\pi f^1_h(u_h)}{1 - \pi + \pi f^1_h(u_h)} \Pi^2_h(u_1, U^2_h(u_1)) = \frac{v_h - c_l}{c_h - c_l} \tag{121}
\]

This system of equations (116) – (121), along with the boundary conditions

\[
F^1_i(c_1) = F^1_h(c_h) = 0, \\
F^2_i(c_1) = \alpha, \\
F^1_i(\bar{u}_1) = F^2_i(\bar{u}_1) = 1, \\
F^1_h(\bar{u}_h) = F^2_h(\bar{u}_h) = 1, \\
(1 - \pi) (v_1 - c_1) = (1 - \pi + \pi f^1_i(c_1 + s)) (v_1 - c_1 - s) + (1 - \pi) \Pi_h(c_1 + s, c_h)
\]

characterize the six unknown functions \( F^1_i, F^2_i, F^1_h, F^2_h, U^1_h \), and \( U^2_h \).

**B General Trading Mechanisms**

In our equilibrium construction, we assumed that buyers offer menus consisting of two contracts—one for high-quality sellers and one for low-quality sellers. In this section, we show that this assumption is without loss of generality. In particular, we consider a game where sellers can send arbitrary messages and buyers offer mechanisms that are deterministic and exclusive—but otherwise unrestricted—mapping the seller’s message into potential terms of trade.\(^{55}\) We prove that the distribution of trades in any equilibrium of this more general setting coincides with that of a game with two-point menus. We prove this within the context of our baseline model, where two buyers face a continuum of sellers.

Intuitively, this result essentially shows that it is impossible for a buyer to screen a seller based on her outside offer. To see why, note that screening is possible only when the payoffs from accepting a given contract differ across types. For example, a seller with a low-quality good gets less utility (compared to one with a high quality good) from accepting a contract which requires her to retain a fraction of the good. However, sellers who differ only in their alternative offers get the same utility from accepting a contract; since trading is exclusive, once they accept the terms of a given contract, their outside offer is irrelevant. This feature rules out the ability to screen sellers along this dimension.

The proof proceeds in two steps. First, we map our environment into the general framework of Martimort and Stole (2002), hereafter MS. This allows us to apply their “delegation principle,” which establishes that any equilibrium of a game with general mechanisms and messages can be achieved by a menu game. Second, we show that equilibrium menus have at most two contracts that are accepted by sellers in equilibrium. Together, these steps imply that a game where buyers offer 2-point menus induces the same equilibrium distribution of trades as a more general game with arbitrary mechanisms and communication.

**Step 1.** We begin by expressing payoffs and strategies using the notation of MS. A contract is defined

\(^{55}\)In a deterministic mechanism, the mapping from the seller’s message to an offer is a deterministic function. Note, however, that buyers can still randomize over different mechanisms.
by a quantity-transfer pair \( d = (x, t) \). The seller’s type is given by \( \theta = (j, \Lambda) \), where \( j \in \{l, h\} \) is the quality of her good and \( \Lambda \subset (1, 2) \) is the set of buyers with whom she is matched. Given a pair of contracts offered by the two buyers, \( d = (d^1, d^2) \), the payoff to a seller of type \( \theta \) is

\[
U(d; \theta) = \max_{i \in A} \ t^i + (1 - x^i) c_j.
\] (122)

When a seller has access to both of the buyers and is indifferent between the contracts they offer, we assume she randomizes, with each buyer being chosen with equal probability. We denote the seller’s contract choice by \( s^i(d; \theta) \), where \( s^1(d; \theta) + s^2(d; \theta) = 1 \), so that the buyer’s payoff can be written

\[
V^i(d; \theta) = (v_j x^i - t^i) s^i(d; \theta).
\] (123)

There is an unrestricted space of messages, denoted \( M \), available to each buyer-seller pair. The strategy space for buyers is the space of all deterministic communication mechanisms. Formally, such a mechanism consists of a mapping \( \hat{\mathcal{d}} : \mathcal{M} \rightarrow \mathcal{D} \) from messages to the set of all contracts \( \mathcal{D} = [0, 1] \times \mathbb{R}^+ \). The set of such mechanisms is represented by \( \mathcal{Y} = \langle \mathcal{D} \rangle^\mathcal{M} \). Each buyer’s strategy \( \sigma \), then, is a distribution over the elements of \( \mathcal{Y} \). A seller’s strategy is a joint distribution over messages sent to each buyer with whom she is matched. The timing of the game is as follows. First, sellers draw their types. Second, each of the buyers simultaneously offers a mechanism to the sellers with whom they are matched. Third, each seller chooses a message to send to each of the buyers with whom she is matched. These choices then induce (potentially a pair of) contracts, with the resulting payoffs given by (122)–(123).

We can now apply the delegation principle from MS (Theorem 1). It states that the distribution of contracts and trades induced by any Perfect Bayesian Equilibrium in the game with mechanisms can be achieved by a game where buyers post menus of contracts and sellers choose their desired contract. Formally, a menu game is one in which each buyer’s strategy is a distribution (possibly random) \( \mu \in \Delta (2^\mathcal{D}) \) over all possible menus \( z \subset \mathcal{D} \). Facing two menus, a seller of type \( j \) proceeds in two steps. First, she chooses a contract from each menu, which is described by a probability distribution \( \chi_j(z_1, z_2; \theta) \in \Delta (z_1 \times z_2) \) over pairs of contracts \( d \in z_1 \times z_2 \). She then chooses one of the two contracts according to the functions \( s^i(\cdot) \) described above.

**Step 2.** The second step, stated formally in the following result, shows that equilibrium menus cannot contain more than two “active” contracts, i.e. ones that are actually traded in equilibrium.

**Proposition 12.** In any equilibrium of the menu game, any menu \( z \) has at most two contracts that are chosen by some seller type in equilibrium.

**Proof.** Without loss of generality, consider an arbitrary menu \( z \) offered by buyer 1 with positive probability in equilibrium, and define \( D_j(z) \) as the set of all contracts in that menu that are chosen by a type \( j \) seller with positive probability, i.e.,

\[
D_j(z) = \{d^1 \in z : \exists d^2 \in z' \in \text{Supp}(\mu) : (d^1, d^2) \in \text{Supp}(\chi_j(z, z')) , s^1(d^1, d^2; (j, \cdot)) > 0\}.
\]

We will show that \( |D_j(z)| = 1 \) for \( j \in \{l, h\} \). The strategy is to show that all elements in \( D_j(z) \) must yield the same utility to type \( j \) sellers and the same payoffs to the buyer, i.e., for all \( (x, t), (x', t') \in D_j(z) \), we must have

\[
t + c_j (1 - x) = t' + c_j (1 - x')
\] (124)

\[
v_j x - t = v_j x' - t',
\] (125)

which implies \( (x, t) = (x', t') \). It is easy to see that the two contracts must offer the same utility to the seller, otherwise she cannot choose both from the same menu with positive probability. To show that they must yield the same payoff to the buyer, consider the offer intended for the type \( l \) seller. Now,
suppose that \((x, t), (x', t') \in D_l(z)\) with \(v_l x - t > v_l x' - t'\). This inequality, combined with (124), implies
\[ v_l (x - x') > t - t' = c_l (x - x') \Rightarrow x > x'. \]
As a result,
\[ c_h (x - x') > c_l (x - x') = t - t' \Rightarrow t + c_h (1 - x) > t' + c_h (1 - x'). \]
This implies that \((x', t') \notin D_h(z)\). Hence, if we eliminate from \(z\) every contract in \(D_l(z)\) except the one that delivers the maximum payoff to the buyer from type \(l\) sellers, the buyer’s payoff strictly increases and the high type seller’s choice is not altered. Therefore, if there is more than one element in \(D_l(z)\), they must all yield the same profits.

Now suppose there exist \((x, t), (x', t') \in D_h(z)\) such that \(v_h x - t > v_h x' - t'\). As before, this implies \(x > x'.\) We then have
\[ t - t' = c_h (x - x') > c_l (x - x') \Rightarrow t + c_l (1 - x) > t' + c_l (1 - x'). \]
Hence, \((x', t') \notin D_l(z)\). Then, as with type \(l\) sellers, eliminating all contracts in \(D_h(z)\) that deliver less than the maximum payoff to the buyer is a profitable deviation. This concludes the proof.

### C Masspoint Equilibria: The Case of \(\phi_1 = 0\)

**Proposition 13.** Suppose \(\phi_1 = 0\). The unique equilibrium of the game is described by the pair of distribution functions, with \(F_l(u_l)\) degenerate at \(v_l\) and \(F_h(u_h)\) satisfying
\[ (1 - \pi + \pi F_h(u_h)) \mu_h \Pi_h(v_l, u_h) = (1 - \pi) \mu_l (v_l - c_l) \]
with Supp\(F_h) = [c_h, c_h + \pi(v_l - c_l)(v_h - c_h)/(v_h - c_l)].\]

**Proof of Proposition 13.** To show that the constructed distributions constitute an equilibrium, we show that there are no profitable deviations. In other words,
\[ \forall (u'_l, u'_h) : \mu_h (1 - \pi + \pi F_l(u'_l)) \Pi_h(u'_l, u'_h) + \mu_l (1 - \pi + \pi F_l(u'_l)) (v_l - u'_l) \leq (1 - \pi) \mu_l (v_l - c_l). \]
We consider two cases:

1. \(u'_h > \max \text{Supp}(F_h) = \bar{u}_h\): In this case, when \(u'_l > v_l\), the profit function is given by
\[ \mu_h \Pi_h(u'_l, u'_h) + \mu_l (v_l - u'_l). \]
Since \(\phi_1 = 0\), the above function is invariant to changes in \(u'_h\) and is strictly decreasing in \(u'_l\). Therefore, its value must be less than its value evaluated at \((u_h, v_l)\), which gives the equilibrium profits. When, \(u'_l \leq v_l\), the profits are given by \(\mu_h \Pi_h(u'_l, u'_h)\), which is decreasing in \(u'_h\), and therefore
\[ \mu_h \Pi_h(u'_l, u'_h) + \mu_l (1 - \pi)(v_l - u'_l) < \mu_h \Pi_h(u'_l, \bar{u}_h) + \mu_l (1 - \pi)(v_l - u'_l). \]
Note that the right-hand side of the above inequality is a linear function of \(u'_l\) whose derivative is given by
\[ \frac{\mu_h v_h - c_h}{v_l - c_l} - \mu_l (1 - \pi) = \frac{\mu_h v_h - c_h}{c_h - c_l} - \mu_l + \mu_l \pi = -\mu_l \phi_1 + \mu_l \pi = \mu_l \pi > 0. \]
Therefore, we must have that
\[
\mu_h \Pi_h (u'_h, \bar{u}_h) + \mu_1 (1 - \pi) (v_1 - u'_1) \leq \mu_h \Pi_h (v_1, \bar{u}_h) = (1 - \pi) \mu_1 (v_1 - c_1)
\]
where the last equality follows from (126).

2. \(u'_h \in [c_h, \bar{u}_h]\). In this case, when \(u'_1 > v_1\), profits are given by
\[
\mu_h \left(1 - \pi + \pi F_1 (u'_h)\right) \Pi_h (u'_h, u'_h) + \mu_1 (v_1 - u'_1) \leq \mu_h \left(1 - \pi + \pi F_1 (u'_h)\right) \Pi_h (v_1, u'_h) = (1 - \pi) \mu_1 (v_1 - c_1)
\]
where the inequality is satisfied since \(u'_1 > v_1\) and the last equality follows from (126).

When \(u'_1 \leq v_1\), profits are given by
\[
\mu_h \left(1 - \pi + \pi F_1 (u'_h)\right) \Pi_h (u'_h, u'_h) + \mu_1 (1 - \pi) (v_1 - u'_1).
\]

The above function is linear in \(u'_1\) and its derivative is given by
\[
\mu_h \left(1 - \pi + \pi F_1 (u'_h)\right) \frac{v_h - c_h}{c_h - c_1} - \mu_1 (1 - \pi) = (1 - \pi) \left(\mu_h \frac{v_h - c_h}{c_h - c_1} - \mu_1\right) + \pi F_1 (u'_h) \frac{v_h - c_h}{c_h - c_1} = \pi F_1 (u'_h) \frac{v_h - c_h}{c_h - c_1} \geq 0.
\]

Therefore, it is maximized at \(u'_1 = v_1\). This establishes that there are no profitable deviations.

To conclude the proof, we show that the equilibrium constructed is the unique equilibrium when \(\phi_1 = 0\).

In order to show uniqueness of equilibrium, it is sufficient to show that, in any equilibrium, \(F_1\) must be degenerate at \(v_1\). When \(F_1\) is degenerate at \(v_1\), from Lemmas 7 and 10, we know that \(F_h\) must be continuous and strictly increasing and therefore it must satisfy (126).

Suppose that \(u_1 \neq v_1\) exists that belongs to the support of \(F_1\). Then the proof of Lemma 11 can be used to show that for values of \(u_1 \neq v_1\), \(F_1\) must have no flat and mass points and consequently equilibrium must exhibit the strict rank-preserving (SRP) property. Now consider any menu for which \(u_1 < v_1\) and a deviation that increases the value of \(u_1\) by a small amount. In this case, \(F_1\) is differentiable and we can write the change in profits from such a deviation as
\[
\mu_1 \pi f_1^+(u_1)(v_1 - u_1) - \mu_1 (1 - \pi + \pi F_1(u_1)) + \mu_h \frac{v_h - c_h}{c_h - c_1} (1 - \pi + \pi F_h(u_h)) = \mu_1 \pi f_1^+(u_1)(v_1 - u_1) - \mu_1 \phi_1 (1 - \pi + \pi F_1(u_1)) > 0
\]
where in the above \(f_1^+\) is the right derivative of \(F_1\) and we have used SRP. The above implies that increasing \(u_1\) must be a profitable deviation which proves the contradiction. The case with \(u_1 > v_1\) is ruled out in a similar fashion. This concludes the proof.

D The Model with Many Types

D.1 Environment and Construction of Equilibrium

We now extend our analysis to the case with an arbitrary, finite number of seller types. We focus our attention on equilibria where all offers are separating menus. We do so for two reasons. First, in the case of \(N = 2\), this region yields some of the most interesting results—such as the non-monotonicity of welfare in \(\pi\)—and we want to confirm that these results are true in a more general setting. Second, in
the equilibrium with all separating menus, the monotonicity constraints are slack ($x_i < x_{i+1}$), which is the most commonly studied case in the mechanism design literature.\textsuperscript{56} We first provide a method for constructing such a separating equilibrium, and then use the constructed equilibrium to demonstrate that the welfare implications from the model with two types extend to the general case of $N > 2$.

Suppose there are $N \geq 2$ types, with buyers and sellers deriving utility $v_i$ and $c_i$, respectively, per unit from a good of type $i \in N \equiv \{1, \ldots, N\}$. The types are ordered so that $v_1 < v_2 < \ldots < v_N$ and $c_1 < c_2 < \ldots < c_N$, and there are gains from trading all types of goods, i.e., $v_i > c_i$ for all $i \in N$. The distribution of types is summarized by the vector $(\mu_1, \ldots, \mu_N)$, with $\sum_{i\in N} \mu_i = 1$. As in our benchmark model, sellers (of all types) are privately informed about the quality of their good and receive two offers (distribution of types is summarized by the vector $\{\mu_1, \ldots, \mu_N\}$, with $\sum_{i\in N} \mu_i = 1$). As in our benchmark model, sellers (of all types) are privately informed about the quality of their good and receive two offers.

**Equilibrium Properties.** The definition of strategies and a (symmetric) equilibrium are identical to those in the model with two types, and hence we omit them for brevity. We begin our analysis, in Lemma 20 below, by establishing that buyers’ offers never distort the quantity traded with the lowest type of seller, and that local incentive constraints always bind “upward,” i.e., equilibrium offers always leave a type $i$ seller indifferent between his contract and the one intended for type $i+1$. As a result, a buyer’s offer can again be summarized by the indirect utilities it delivers to each type $i \in N$.

**Lemma 20.** For almost all equilibrium menus:

1. There is full trade with the lowest type, so that $x_1 = 1$, and the local incentive constraints are binding upward, so that
   \[ t_1 + c_1 (1 - x_1) = t_{i+1} + c_1 (1 - x_{i+1}) \text{ for all } i = 1, 2, \ldots, N - 1; \]

2. Each menu can be summarized by a utility vector $u = (u_1, \ldots, u_N)$ with $u_i \geq c_i \forall i$ and
   \[ 1 \geq \frac{u_N - u_{N-1}}{c_N - c_{N-1}} \geq \cdots \geq \frac{u_2 - u_1}{c_2 - c_1} \geq 0, \tag{127} \]

   with the corresponding quantities and transfers given by
   \[ x_1 = 1, \quad x_i = 1 - \frac{u_i - u_{i-1}}{c_i - c_{i-1}}, \quad i = 2, 3, \ldots, N \tag{128} \]
   \[ t_1 = u_1, \quad t_i = u_i - \frac{c_i}{c_i - c_{i-1}} (u_i - u_{i-1}), \quad i = 2, 3, \ldots, N. \]

This proof of Lemma 20 is a direct extension of the proof of Lemma 1, and hence is omitted for brevity. Given the results, we can recast each buyer’s problem in terms of the utility vector $u$. In particular, given a family of marginal distributions $F_i(u_i)$ for $i \in N$, each buyer chooses a vector $u$ to solve

\[
\max_{u_i \geq c_i} \sum_{i=1}^{N} \mu_i \left(1 - \pi + \pi F_i(u_i)\right) \Pi_i(u_{i-1}, u_i) \tag{129}
\]

subject to the monotonicity constraints in (127), where (in a slight abuse of notation) profits per trade with a seller of quality $i$ are given by

\[
\Pi_1(u_1) = v_1 - u_1, \quad \Pi_i(u_{i-1}, u_i) = v_i - \frac{v_i - c_{i-1}}{c_i - c_{i-1}} u_i + \frac{v_i - c_i}{c_i - c_{i-1}} u_{i-1}, \quad \text{for all } i = 2, \ldots, N. \tag{130}
\]

\textsuperscript{56}See, e.g., Fudenberg and Tirole (1991).
The program in (129) resembles a standard mechanism design problem, where the binding incentive constraints are substituted into the profit functions in (130). The monotonicity constraints in (127) are necessary to ensure that local incentive compatibility implies global incentive compatibility.

We now formally define a separating equilibrium, provide a characterization and a method for constructing such equilibria, and then use numerical examples to study their normative properties.

**Definition 2.** An equilibrium is separating if the utility vector $u$ associated with any equilibrium menu solves the relaxed problem of maximizing the objective in (129) ignoring the monotonicity constraints in (127).

As a first step, in the conjectured equilibrium, one can use an induction argument to extend Proposition 1, establishing that all the distributions $F_i$ are continuous with connected support. Since the profit function is strictly supermodular, any separating equilibrium must satisfy the strict rank-preserving property. The following proposition summarizes.

**Proposition 14.** If $\phi_1 = 1 - \frac{\mu_2 v_2 - c_2}{\mu_1 c_2 - c_1} \neq 0$, then, in any symmetric separating equilibrium,

1. For all $i \in N$, $F_i(\cdot)$ has a connected support and is continuous.

2. There exists a sequence of strictly increasing real-valued functions $(U_i(u_1))_{i=1}^N$ such that the utility vector associated with any equilibrium menu $z$ satisfies:

$$u(z) = (u_1(z), U_2(u_1(z)), U_3(u_1(z)), \cdots, U_N(u_1(z))).$$

As in the model with two types, Proposition 14 greatly simplifies the construction of separating equilibria: it implies that we only need to characterize the distribution of offers to the lowest type, $F_1(u_1)$, together with the sequence of functions $(U_i(u_1))_{i=2}^N$. The equilibrium distribution of utilities can then be derived from the fact that all types have the same ranking across equilibrium menus, i.e., $F_i(U_i(u_1)) = F_1(u_1)$ for all $i = 2, \ldots, N$.

**Equilibrium construction.** We now illustrate how to construct a separating equilibrium. Differentiability of the profit function in (129) implies that any separating equilibrium must satisfy

$$\frac{\pi f_i(U_i(u_1))}{1 - \pi + \pi F_i(U_i(u_1))}\Pi_1(u_1) = \phi_i,$$

$$\frac{\pi f_i(U_i(u_1))}{1 - \pi + \pi F_i(U_i(u_1))}\Pi_1(U_{i-1}(u_1), U_i(u_1)) = \phi_i \quad \text{for all } i = 2, \ldots, N,$$

where $\phi_i$, the marginal cost of increasing the utility of a seller of type $i$, is given by

$$\phi_1 = 1 - \frac{\mu_2 v_2 - c_2}{\mu_1 c_2 - c_1},$$

$$\phi_i = \frac{v_i - c_{i-1}}{c_i - c_{i-1}} - \frac{\mu_{i+1} v_{i+1} - c_{i+1}}{\mu_i c_{i+1} - c_i}, \quad \text{for all } i = 2, \cdots, N - 1,$$

$$\phi_N = \frac{v_N - c_{N-1}}{c_N - c_{N-1}}.$$

Equation (132) implies that $F_1$ must satisfy

$$\frac{\pi f_1(u_1)}{1 - \pi + \pi F_1(u_1)} = \frac{\phi_1}{v_1 - u_1}.$$ 

---

57This proposition relies on the assumption that the marginal cost of transfers to the lowest type net of any benefits arising from binding incentive constraints, $\phi_1$, is non-zero. As in the two-type case, this assumption is required to show that equilibrium distributions do not have mass points.
Since the strict rank-preserving property implies that each $U_i$ must satisfy $F_i(U_i(u_1)) = F_1(u_1)$, it must be the case that $U_i'(u_1)f_1(U_i(u_1)) = f_1(u_1)$. Substituting this result into (133) implies that the equilibrium functions $U_i$ must satisfy the set of differential equations:

$$U_i'(u_1) = \frac{\phi_i \Pi_i(U_{i-1}(u_1), U_i(u_1))}{\phi_i} \frac{v_1 - u_1}{v_1} \quad \text{for all } i = 2, \ldots, N. \quad (135)$$

The system of differential equations (134) and (135) are ordinary first order differential equations; to complete the characterization, we need only provide the appropriate boundary conditions. As in the two-type model, these conditions depend critically on the marginal costs, $\{\phi_1, \ldots, \phi_N\}$, and are closely tied to the outcome under monopsony. The following result shows that the solution to a monopsonist’s problem can be represented in the form of a threshold type.

**Lemma 21.** Let $J$ denote the largest integer $i \in \{1, 2, \ldots, N\}$ such that

$$\sum_{i=1}^{J-1} \mu_i \phi_i < 0,$$

with $J = 1$ if $\sum_{i=1}^{k} \mu_i \phi_i > 0$ for all $k \in \{1, 2, \ldots, N\}$. The solution to a monopsonist’s problem is to set $u_i = c_J$ for $i \leq J$ and $u_i = c_i$ for $i > J$.

Intuitively, the accumulated marginal cost of trading with the first $J$ types is negative ($\sum_{i=1}^{J-1} \mu_i \phi_i < 0$), so they are pooled. In contrast, for the remaining types, the information rents outweigh the potential gains, so the monopsonist chooses not to trade with them. The next result links this threshold $J$ to the best and worst menu when $\pi > 0$.

**Lemma 22.** Let $J$ be as defined in Lemma 21. Then, in any equilibrium, the best menu has $u_i = u_J$ for $i < J$, and the worst menu has $u_i = c_i$ for all $i \geq J$.

To see the intuition, note that the best menu trades with probability 1, i.e., attracts all captive and noncaptive sellers. Therefore, it cannot be profitable for that menu to separate types that a monopsonist finds profitable to pool; if $u_i < u_J$ for some $i < J$, then increasing $u_i$ has no effect on the probability or composition of trades but yields strictly higher profits (because the effective marginal cost of increasing $u_i$ is negative). Similarly, it cannot be profitable for the worst menu to give any surplus to the types that the monopsonist finds optimal to shut out completely; if such a menu offers more than $c_i$ to any type $i > J$, the buyer can raise her profits simply by lowering that utility.

The system of differential equations (134)-(135), along with the boundary conditions described in Lemma 22, describe necessary conditions for any separating equilibrium. By the Picard-Lindelöf theorem, it has a unique solution. In Appendix D.2.5, we provide analytical expressions for this solution. To ensure that this solution is an equilibrium, one need only verify that the monotonicity constraints (127) are satisfied for every $u_1 \in \text{Supp}[F_1]$.

Finally, we solve two numerical examples using the method described above. The two cases both have $N = 4$, but differ in the marginal cost vector, $\{\phi_1, \ldots, \phi_N\}$. In the first case, $J = 1$, so the monopsonist only trades with the lowest type. In the second case, $J = 2$. We use these cases to demonstrate the robustness of the welfare results in section 5.2. In Figure 10, we plot expected trade for types 2 through 4 (recall that $x_1 = 1$ always) as a function of $\pi$. They show a non-monotonic relationship between expected trade and competition. In the first case (left panel), in which the monopsonist only trades with type 1, trade by all three types is hump-shaped. This is analogous to the case with $\phi_1 > 0$ in the two-type model. In

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58 For brevity, we ignore the non-generic case in which the inequality in (136) is satisfied with equality.

59 For both cases, we assume a uniform distribution $\mu_i = 0.25$ for all $i$, with valuations $c_1 = 1, 2, 3, 4$ and $v_1 = c_i \delta + 0.5$. In case 1, $\delta = 1.2$ and in case 2, $\delta = 1.3$. In each case, we solve the system (134)-(135) and verify that the monotonicity constraints are satisfied.
the second case (right panel), however, trade with one of the types (type 2) is monotonically decreasing in $\pi$. This is similar to the case with $\phi_1 < 0$ in the two-type model. In both cases, these patterns imply that ex-ante welfare is maximized at $\pi < 1$.

![Figure 10: Expected trade and competition when $N=4$ and $J=1$ (left panel) or $J=2$ (right panel).](image)

D.2 Proofs

D.2.1 Proof of Lemma 20

This proof is a direct extension of the proof of Lemma 1, and hence is omitted for brevity.

D.2.2 Proof of Proposition 14

To show the strict rank-preserving property, we first show that $F_j$'s are continuous and strictly increasing. The argument for this claim is inductive.

Step 1: $F_N$ is strictly increasing and continuous.

$F_N$ is strictly increasing. Suppose, towards a contradiction, that there is an interval $[u''_N, u''_N]$ where $F_N$ is constant and takes a value between 0 and 1. Without loss of generality, we can assume that $u''_N$ belongs to some contract that is offered in equilibrium. Let one such menu be given by $u'' = (u''_1, \cdots, u''_N)$. Given our assumption that the equilibrium is separating, this menu must maximize $\sum_{i=1}^N \mu_i (1 - \pi + \pi F_i (u_i)) \Pi_i (u_{i-1}, u_i)$ over the set of menus that are subject to the participation constraints. Now consider a menu given by $(u''_1, \cdots, u''_{N-1}, u''_N - \varepsilon)$ for a small $\varepsilon$. Since $u''_N > u'_N \geq c_N$, this menu satisfies the participation constraint. Moreover, this menu keeps the fraction of noncaptive $N$ types constant while increasing profits per $N$-th type, thus yielding higher profits, a contradiction.

$F_N$ is continuous. Suppose, towards a contradiction, that $F_N$ has a mass point at $\hat{u}_N$. Let $u = (u_1, \cdots, u_{N-1}, u_N)$ be an arbitrary equilibrium menu with its $N$-th element given by $\hat{u}_N$. Note that we must have $\Pi_N (u_{N-1}, \hat{u}_N) \leq 0$ and $\hat{u}_N = c_N$. The fact that $\Pi_N (u_{N-1}, \hat{u}_N) \leq 0$ is immediate, since otherwise a small increase in $\hat{u}_N$ would attain a higher level of profits. Additionally, if $\hat{u}_N > c_N$, then a small decrease in $\hat{u}_N$ would attain higher profits. Such a change increases profits because either $\Pi_N < 0$—in which case this change decreases the probability that an $N$ type accepts the offer discretely—or $\Pi_N = 0$—in which case this change makes profits per $N$ type strictly positive.
Non-positivity of profits, together with \( \hat{u}_N = c_N \), implies that
\[
\frac{\nu_N - \nu_N - c_{N-1} c_N}{c_N - c_{N-1}} + \frac{\nu_N - c_N}{c_N - c_{N-1}} u_{N-1} \leq 0 \Rightarrow \frac{\nu_N - c_N}{c_N - c_{N-1}} u_{N-1} \leq \frac{\nu_N - c_N}{c_N - c_{N-1}} c_{N-1} \Rightarrow u_{N-1} \leq c_{N-1}.
\]
This inequality, together with the participation constraint, \( c_{N-1} \leq u_{N-1} \), implies that \( u_{N-1} \) must equal \( c_{N-1} \) and \( \Pi_N = 0 \). That is, any menu \( u \) with \( \hat{u}_N \) as its N-th element must also satisfy \( u_{N-1} = c_{N-1} \), so that \( F_{N-1} \) must also have a mass point at \( c_{N-1} \). Repetition of this argument implies that any menu containing a mass point at \( \hat{u}_N \) must also satisfy \( u_j = c_j \), and thus \( F_j \) must have a mass point at \( c_j \). However, then a small increase in \( u_1 \) from \( u_1 = c_1 \) must increase profits, as \( F_1 \) puts a mass at \( c_1 \) and profits from type 1 sellers are positive. This yields the necessary contradiction.

**Step 2:** If \( \{F_k\}^{N}_{k=j+1} \) are strictly increasing and continuous, then \( F_j \) must have the same properties.

To prove this claim, we first prove the following lemma:

**Lemma 23.** Suppose that, for some \( j \leq N - 1 \), the distributions \( \{F_k\}^{N}_{k=j+1} \) are continuous and strictly increasing. Then there exists a sequence of strictly increasing and continuous functions \( \{U_{k,j}(u_j)\}^{N}_{k=j+1} \) such that for any menu \( \hat{u} \) offered in equilibrium with its \( j \)-th element given by \( \hat{u}_j \)
\[\left(\hat{u}_{j+1}, \ldots, \hat{u}_N\right) = \left(U_{j+1,j}(\hat{u}_j), \ldots, U_{N,j}(\hat{u}_j)\right).\]

**Proof.** We prove this claim by induction. For any value of \( u_{N-1} \), let \( U^+_N(u_{N-1}) \) be the set of values of \( u_N \) such that equilibrium menus exist with \( (N-1) \)-th and \( N \)-th elements given by \( (u_{N-1}, u_N) \).

We first show that \( U^+_N(u_{N-1}) \) is a strictly increasing function. Using exactly the same arguments as in the two-type case, it is straightforward to show that: (i) \( U^+_N(u_{N-1}) \) must be a strictly increasing correspondence; and (ii) if \( u, u' \in U^+_N(u_{N-1}) \), then \( [u, u'] \subseteq U^+_N(u_{N-1}) \). These results are direct implications of strict supermodularity of the function \( \mu_N(1 - \pi + \pi F_N(u_N)) \Pi_N(u_{N-1}, u_N) \) and the strict monotonicity of \( F_N \).

Now suppose that for some \( \hat{u}_{N-1} \), \( U^+_N(\hat{u}_{N-1}) \) is a correspondence and so contains an interval given by \([u', u'']\). Then
\[
\Pr(u_{N-1} = \hat{u}_{N-1}) = \int_{\{(u_{N-2}, \hat{u}_{N-1}, u_N)\in\text{Supp}(\Phi)\}} d\Phi \geq F_N(u'') - F_N(u') > 0,
\]
where the last inequality follows from the fact that \( F_N \) is strictly increasing. This inequality implies that \( F_{N-1} \) has a mass point at \( \hat{u}_{N-1} \), in contradiction with the assumption that \( F_{N-1} \) is continuous. Hence, \( U^+_N(u_{N-1}) \) must be a single-valued function.

One can also adapt our arguments from the two-type case to show that \( U^+_N(u_{N-1}) \) is strictly increasing. If it were constant on an interval, then \( F_N \) must have a mass point, contradicting the continuity of \( F_N \). Thus, \( U^+_N(u_{N-1}) \) is a strictly increasing function and we may write profits from the \( N \)-th type as function of \( u_{N-1} \) only. Let this function be given by \( \Pi^+_N(u_{N-1}) \).

Next, let \( U^+_N(u_{N-2}) \) be defined in a similar fashion as above. Since the profit function
\[
\mu_{N-1}(1 - \pi + \pi F_{N-1}(u_{N-1})) \Pi_{N-1}(u_{N-2}, u_{N-1}) + \Pi^+_N(u_{N-1})
\]
is strictly supermodular and \( F_{N-1} \) and \( F_{N-2} \) are strictly increasing and continuous, \( U^+_N(u_{N-1}) \) must be a strictly increasing, single-valued function. Exact repetition of this argument implies that for all \( k \in \{j, \ldots, N - 1\} \), \( U^+_k \) is a strictly increasing function. Therefore, we must have that
\[
U_{k,j}(\hat{u}_j) = U^+_k \left( U^+_{k-1} \left( \ldots \left( U^+_{j+1}(\hat{u}_j) \right) \right) \right)
\]
for all \( k \in \{j + 1, \ldots, N\} \), and this concludes the proof.
We now return to proving step 2 of the induction argument.

$F_j$ is strictly increasing. Suppose, by way of contradiction, that $F_j$ has a flat over an interval $[u'_j, u''_j]$. Much as in Lemma 11, we prove that if $F_j$ is flat on the interval $[u'_j, u''_j]$, then the marginal benefit of delivering one additional unit of surplus to type $j + 1$ (incorporating the impact on all types $i > j + 1$) changes with $u_j \in [u'_j, u''_j]$. This fact allows us to show alternative menus with higher levels of profits than the conjectured equilibrium level must exist.

To see this, first let $U^+_{j+1}(u_j)$ be the correspondence defined in the proof of Lemma 23. By our induction assumption and Lemma 23, profits from types $\{j + 1, \ldots, N\}$ can be written as

$$
\mu_{j+1} \left( 1 - \pi + \pi F_{j+1}(u_{j+1}) \right) \Pi_{j+1}(u_j, u_{j+1}) + \Pi^+_{j+2}(u_{j+1})
$$

where $\Pi^+_{j+2}(u_{j+1})$ are equilibrium profits constructed by applying $U_{k,j+1}$ as defined in Lemma 23. Note that these profits are strictly supermodular in $(u_j, u_{j+1})$, and, as a result, $U^+_{j+1}(u_j)$ is a strictly increasing correspondence. Additionally, since $F_j$ is flat over the interval $[u'_j, u''_j]$, we must have that $U^+_{j+1}(u'_j)$ and $U^+_{j+1}(u''_j)$ must have a common element (as in the proof of Lemma 11). Let $\bar{u}_{j+1}$ be this common element.

Let $u'$ be an equilibrium menu with $j$-th element given by $u'_j$ and $(j + 1)$-th element given by $\bar{u}_{j+1}$ and $u''$ be an equilibrium menu with $j$-th element given by $u''_j$ and $j + 1$-th element given by $\bar{u}_{j+1}$. Note that a perturbation of $u''$ which increases $u''_j$ by a small amount must not increase profits. Similarly, a perturbation of $u''$ which decreases $u''_j$ by a small amount must not increase profits. Since $F_j$ is flat on $[u'_j, u''_j]$, non-positivity of these two perturbations imply

$$
- \mu_j F_j(u'_j) \frac{v_j - c_{j-1}}{c_j - c_{j-1}} + \mu_{j+1} F_{j+1}(\bar{u}_{j+1}) \frac{v_{j+1} - c_{j+1}}{c_{j+1} - c_j} = 0.
$$

As a consequence, profits obtained from any menu $\hat{u}$, which is the same as $u'$ except at its $j$-th element and has $j$-th element equal to $u_j \in [u'_j, u''_j]$, must yield the same profits as $u'$.

We now show that a perturbation from some such $\hat{u}$ must strictly increase profits. In particular, consider a perturbation from $\hat{u}$ which increases $u_{j+1} = \bar{u}_{j+1}$ by a small amount, $\varepsilon$. Since $F_{j+1}$ is strictly increasing and continuous, the change in profits from this perturbation is given by

$$
\mu_{j+1} f_{j+1}(\bar{u}_{j+1}) \Pi_{j+1}(u_j, \bar{u}_{j+1}) + \mu_{j+1} \left( 1 - \pi + \pi F_{j+1}(\bar{u}_{j+1}) \right) \frac{v_j - c_j}{c_{j+1} - c_j} + \frac{d}{du_{j+1}} \Pi^+_{j+2}(\bar{u}_{j+1}).
$$

Since $f_{j+1}(\bar{u}_{j+1}) > 0$ and $\Pi_{j+1}$ is linear in $u_j$ the expression in (138) must be non-zero for some $u_j \in (u'_j, u''_j)$. This implies some menu can strictly raise profits above the conjectured equilibrium level and is a contradiction. Thus, $F_j$ cannot have a flat.
consider two different perturbations from \( u \) where we perturb elements \( k \) through \( j \) according to
\[
\begin{align*}
u^- &= (v_1, \ldots, \hat{u}_{k-1}, \hat{u}_k - \epsilon, \ldots, u'_j - \epsilon, u'_{j+1}, U_{j+2,j+1}(u'_{j+1}), \ldots, u_{N,j+1}(u'_{j+1})), \\
u^+ &= (v_1, \ldots, \hat{u}_{k-1}, \hat{u}_k + \epsilon, \ldots, u'_j + \epsilon, u''_{j+1}, U_{j+2,j+1}(u''_{j+1}), \ldots, u_{N,j+1}(u''_{j+1})).
\end{align*}
\]

For small \( \epsilon \), the change in the profits from the above perturbations are, respectively, given by
\[
\begin{align*}
\mu_k(1 - \pi + \pi F_k^-(\hat{u}_k)) &\frac{v_k - c_{k-1}}{c_k - c_{k-1}} + \mu_{k+1}(1 - \pi + \pi F_{k+1}^-(\hat{u}_{k+1})) + \cdots + \mu_j(1 - \pi + \pi F_j^-(\hat{u}_j)) \\
-\mu_j+1(1 - \pi + \pi F_j^-(u'_j)) &\frac{v_{j+1} - c_{j+1}}{c_{j+1} - c_j}, \\
-\mu_k(1 - \pi + \pi F_k^+(\hat{u}_k)) &\frac{v_k - c_{k-1}}{c_k - c_{k-1}} - \mu_{k+1}(1 - \pi + \pi F_{k+1}^+(\hat{u}_{k+1})) + \cdots + \mu_j(1 - \pi + \pi F_j^+(\hat{u}_j)) \\
+\mu_{j+1}(1 - \pi + \pi F_{j+1}^+(u''_{j+1})) &\frac{v_{j+1} - c_{j+1}}{c_{j+1} - c_j}.
\end{align*}
\]

Since the distributions \( F_i \) are well behaved above and below each \( \hat{u}_i \), the strict rank preserving property implies \( F^-_i(\hat{u}_i) = F_{j+1}(u'_j) \), and \( F^+_i(\hat{u}_i) = F_{j+1}(u''_{j+1}) \) for all values of \( i \leq j \). We may then write the change in profits from the above perturbations, respectively, as
\[
\begin{align*}
(1 - \pi + \pi F_k^-)(\hat{u}_k) \sum_{i=k}^j \mu_i \phi_i, \\
-(1 - \pi + \pi F_k^+)(\hat{u}_k) \sum_{i=k}^j \mu_i \phi_i.
\end{align*}
\]

Since \( k \) is the highest index below \( j \) for which \( \phi_k \neq 0 \), one of the above expressions must be positive. Therefore, one of the constructed menus increases profits, yielding a contradiction. The claim that equilibrium is strictly rank-preserving then follows immediately from Lemma 23.

\section*{D.2.3 Proof of Lemma 21}

The monopsonist maximizes
\[
\mu_1(v_1 - u_1) + \sum_{i=2}^{N} \mu_i \left[ v_i - \frac{v_i - c_{i-1}}{c_i - c_{i-1}} u_i + \frac{v_i - c_i}{c_i - c_{i-1}} u_{i-1} \right] = \sum_{i=1}^{N} \mu_i (v_i - \phi_i u_i)
\]
subject to the monotonicity constraint
\[
1 \geq \frac{u_n - u_{n-1}}{c_n - c_{n-1}} \geq \cdots \geq \frac{u_{i+1} - u_i}{c_{i+1} - c_i} \geq \frac{u_i - u_{i-1}}{c_i - c_{i-1}} > 0.
\]

Given the linearity in payoffs and constraints, the solution to this problem is a single price offer, i.e., \( u_i = c_j \), \( i \leq J \) and \( u_i = c_i \) for \( i > J \) for some \( J \in \{1, 2, \ldots, N\} \); see arguments in Myerson (1985b) and Samuelson (1984). To see why \( J \) must be the largest integer such that \( \sum_{i=1}^{J-1} \mu_i \phi_i < 0 \), suppose otherwise, i.e., \( \exists \ k < J \) such that \( \sum_{i=1}^{k-1} \mu_i \phi_i < 0 \) and the monopsonist sets \( u_i = c_k \) for \( i \leq k \) and \( u_i = c_j \) for \( i > k \). Then, a deviation which increases all \( u_i \) for \( i < J \) by \( \epsilon \) changes profits by \( -\epsilon \sum_{i=1}^{J-1} \mu_i \phi_i > 0 \).
D.2.4 Proof of Lemma 22

To show that the best equilibrium menu satisfies $u_i = u_j$ for $i < J$, suppose by way of contradiction that for some $i < J$, $u_i < u_j$. The monotonicity constraint implies $u_j > u_{i-1}$; if $u_j = u_{i-1}$, then we must have $u_i = u_{i-1}$ for all $i < J$. Now, consider an alternative menu that increases all the utilities of types below $J$ by $\varepsilon$. The probability of trade with any type does not change (since this is already the best menu), the change in profits is given by $\varepsilon \sum_{i=1}^{J-1} \mu_i \phi_i$, which is strictly positive by the definition of $J$ in (136).

To show that the worst equilibrium menu satisfies $u_i = c_i$ for $i \geq J$, suppose by way of contradiction that $u_{J+k} > c_{J+k}$ for some $k \geq 0$. This inequality, together with repeated application of the monotonicity constraint, implies that $u_i > c_i$ for all $i \leq J + k$. Now consider an alternative menu that lowers the utility of all types below and including $J + k$ by $\varepsilon$. This does not change the probability of trade as the original menu is the worst menu. However, the change in profits from captive types is $\varepsilon \sum_{i=1}^{J+k} \mu_i \phi_i$, which is positive by the definition of $J$ in (136). \hfill \blacksquare

D.2.5 The Solution to the System of ODEs in (135)

The general solution to this system of equations depends on the sign of the profits from the lowest types, $v_1 - u_i$. From (134), this profit is positive when $\phi_1 > 0$, and negative when $\phi_1 < 0$. In what follows, we assume that the sequence $\gamma_i = \frac{v_i - c_{i+1}}{c_i - c_{i+1}} \phi_i$ takes on different values for all $i \geq 2$, i.e., $\gamma_i \neq \gamma_j$.\footnote{While it is possible to provide the general solution of the ODEs, this assumption greatly simplifies the formulation.} We thus have the following general solution:

$$U_i = \sum_{k=0}^{i} a_{k,i} (|v_1 - u_1|)^{\gamma_k}$$

with

$$\gamma_0 = 0, \gamma_1 = 1$$

where

$$a_{0,i} = \frac{v_i (c_i - c_{i-1})}{v_i - c_{i-1}} + \frac{v_i - c_i}{v_i - c_{i-1}} a_{0,i-1}$$

$$a_{k,i} = \frac{v_i - c_i}{v_i - c_{i-1}} \frac{\gamma_i}{\gamma_i - \gamma_{k-1}} a_{k-1,i}$$

with

$$a_{0,1} = v_1$$

$$a_{1,1} = \text{sgn}(v_1 - u_1)$$

where $\text{sgn}$ is 1 if its argument is positive and $-1$ when it’s argument is negative.

In the above formulation, the variables $(a_{i,i})_{i=2}^{N}$ are unknown and have to be determined by the boundary conditions in Lemma 22. To do this, for any value of $u_1 = \min \text{Supp}(F_1)$, we can use equation (134) to solve for $F_1$, with the boundary condition that $u_1$. We can then find the value of $\pi_1$, i.e., the upper bound of the support of $F_1$, using $F_1(\pi_1) = 1$. We refer to this value as $\tilde{u}_1(u_1)$ as a function of $u_1$. The boundary conditions then are given by:

$$U_1(u_1) = c_J, \ldots, U_N(u_1) = c_N$$

$$U_2(\tilde{u}_1(u_1)) = \tilde{u}_1(u_1), \ldots, U_1(\tilde{u}_1(u_1)) = \tilde{u}_1(u_1)$$

The above is a system of $N - J + 1 + J - 1 = N$ equations with $N$ unknowns given by $a_{i,i}^{N}_{i=2}$ and $u_1$. Solving this system of equations determines the equilibrium.