The Tail that Keeps the Riskless Rate Low

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\textsuperscript{1}NYU \quad \textsuperscript{2}NYU Stern
Government bond yields have fallen since 2008-09 and have remained low

Our story:

Great Recession
⇓
Change in beliefs about tail risk
⇓
**Persistent** fall in returns on safe, liquid assets
Key Ingredients

- **Main idea:**
  No one knows the true distribution of aggregate shocks
  → **Re-estimate beliefs as new data arrives**

- **Estimation of beliefs:**
  → **Non-parametric** approach: **tail risk** vs uncertainty
  → Use observed macro data, **empirical discipline**

- **Tail events:** (e.g. the Great Recession)
  → **Large changes** in beliefs, in tail probabilities
  → Changes are **long-lived**, even if the underlying shocks are iid

- **Economic environment:**
  Neoclassical production economy with **liquidity** constraints

- **Quantitative results:**
  → **Large and persistent drop** in riskless rates (1.45%)
  → Consistent with evidence from option markets
Belief formation
• Consider an iid shock, $\phi_t$, with unknown distribution $g$

• Information set: finite history of shock realizations $\{\phi_{t-s}\}_{s=0}^{n_t-1}$

• **Goal:** a flexible specification that can capture tail risk

• We use a non-parametric estimator: the Gaussian kernel density

$$
\hat{g}_t(x) = \frac{1}{n_t \kappa} \sum_{s=0}^{n_t-1} \Omega\left(\frac{x - \phi_{t-s}}{\kappa}\right)
$$
Tail events and beliefs: An example

Tail events → large changes in tail risk (hump on left)
Persistence of belief changes

Exercise I: Simulate future time paths drawing from updated distribution, \( \hat{g}_t \)

- **Beliefs are martingales:** \( \mathbb{E}_t[\hat{g}_{t+j}|\mathcal{I}_t] \approx \hat{g}_t \) \( \rightarrow \) **Persistence**

Exercise II: Simulate future time paths without tail events, re-estimate beliefs

- Beliefs eventually revert, but the pace is very slow

![Histogram and line graph showing changes in belief distribution over time](image)
Economic Model
Model

- **Production:** \( Y_t = K_t^\alpha N_t^{1-\alpha} \)
  - Aggregate shocks to capital 'quality': \( K_t = \phi_t \hat{K}_t \)
  - Law of motion \( \hat{K}_{t+1} = K_t (1 - \delta) + I_t \)

- **Preferences:**
  - Representative HH with stochastic discount factor \( M_t \)
Model

- **Production:** \( Y_t = K_t^\alpha N_t^{1-\alpha} \)
  - Aggregate shocks to capital ‘quality’: \( K_t = \phi_t \hat{K}_t \quad \phi_t \sim iid \ g(\cdot) \)
  - Law of motion \( \hat{K}_{t+1} = K_{t}(1 - \delta) + I_t \)

- **Preferences:**
  - Representative HH with stochastic discount factor \( M_t \)

- **Role of Liquidity:**
  - Opportunity to invest in an intra-period project: payoff \( H(X_t) - X_t \)
  - Liquidity constraint: \( X_t \leq B_t + \eta \phi_t \hat{K}_t \), where \( B_t \equiv \) riskfree bonds

- **Beliefs:**
  - Distribution \( g \) unknown to all agents
  - At each \( t \), observe \( \{\phi_1, \ldots, \phi_t\} \)
  - Gaussian kernel density estimator \( \rightarrow \hat{g}_t \)

\[ \Rightarrow \quad \frac{1}{R_t^f} = E_t [M_{t+1}(1 + Liq_{t+1})] \]
Quantitative Results
Measurement and Calibration

Aggregate shock:

\[
\phi_t = \frac{K_t}{\hat{K}_t} = \frac{\text{Effective capital}}{\text{Yesterday's effective capital} + \text{Investment}}
\]

Data: Non-financial assets of US Corporate Business (Flow of Funds)

- Commercial real estate (\(\sim 55\%\)), equipment and software
- Market value \(\rightarrow\) Effective capital
- Historical cost \(\rightarrow\) Investment \(\Rightarrow\) Direct measure of \(\phi\)

\[
\phi_t = \frac{K_t}{\hat{K}_t} = \left( \frac{P^k_t K_t}{P^k_{t-1} \hat{K}_t} \right) \left( \frac{PINDX^k_t}{PINDX^k_{t-1}} \right)
\]

Calibration:

- Preferences: Risk aversion = 0.5, Frisch = 2
- Liquidity: \(R^f = 0.02\), pledgability of capital = 0.16, \(H'(X) = \zeta/\sqrt{X}\)
Large negative shocks → Large (and persistent) increase in tail risk
Beliefs:

1. Start at ‘steady state’ of \( \hat{g}_{2007} \) (estimated using 1950-2007 data)

2. Feed in the actual shocks from 2008-09 and estimate \( \hat{g}_{2009} \)

\[
(\phi_{2008}, \phi_{2009}) = (0.93, 0.84)
\]

Exercises:

1. Baseline: simulate time paths drawing from \( \hat{g}_{2009} \), plot mean responses

2. No more crisis: simulate paths drawing from \( \hat{g}_{2007} \), plot mean responses
Tail event + Learning → Persistent Fall $R^f$

- **Model (learning):** unknown $g$, beliefs updated as new data arrives
- **Model (no learning):** $g = \hat{g}_{2009}$ known from the beginning
- **Data:** 1-year nominal Treasury yield less expected inflation

Graph showing the change in the riskless rate over time from 2010 to 2040, with trends for the model with and without learning, and actual data points.
## Model vs Data: Long-run changes

<table>
<thead>
<tr>
<th>Riskless rate</th>
<th>Change, %</th>
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<tbody>
<tr>
<td><strong>Model</strong></td>
<td>-1.45</td>
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<td>1-year real rate</td>
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Model: Average in stochastic steady states under $\hat{g}_{2009}$ minus the one under $\hat{g}_{2007}$

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Model: Average in stochastic steady states under $\hat{g}_{2009}$ minus the one under $\hat{g}_{2007}$


### Role of Liquidity:

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Almost all of the drop in $R^f$ comes from the interaction of tail risk and liquidity

Increase in tail risk $\Rightarrow$ Liquidity from capital lower and riskier $\Rightarrow$ bonds become more valuable
What about equity valuations and options prices?

Interpret equity as a levered claim on the value of the representative firm

Returns and valuations:

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<td>1.39</td>
<td>3.83</td>
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<tr>
<td>In Equity/Capital</td>
<td>0.01</td>
<td>0.22</td>
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Higher tail risk does not imply a large fall in equity valuations
What about equity valuations and options prices?

Interpret equity as a levered claim on the value of the representative firm

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Higher tail risk does not imply a large fall in equity valuations

Tail risk indicators:

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</tr>
</thead>
<tbody>
<tr>
<td>Third moment $E^Q(R^e - \bar{R}^e)^3$</td>
<td>−0.002</td>
<td>−0.002</td>
</tr>
<tr>
<td>$Pr^Q(R^e - \bar{R}^e \leq -0.30)$</td>
<td>0.022</td>
<td>0.015</td>
</tr>
</tbody>
</table>

Expectations and probabilities are under the risk-neutral measure.

Option prices show increase in tail risk
What if there are no more crisis?

- **With crises:** Draw future shocks from $\hat{g}_{2009}$ (benchmark)
- **No more crises:** Draw future shocks from $\hat{g}_{2007}$

**Long-lived effects even if crises never occur again**
• Obviously, no one knows the true distribution of shocks

• New data permanently reshapes our assessment of macro risks

• Tail events have long-lived effects on beliefs as data on tail events is scarce

• A new perspective on the persistent drop of riskless rates
Appendix
Contribution to the Literature

Low interest rates:

- Hall (2017), Barro et al. (2014), Bernanke et al. (2011), Carvalho et al. (2016), Caballero et al. (2016), Bigio (2015) and Del Negro et al. (2017)
  - *We add*: new mechanism, acting through belief revisions

Belief-driven business cycles

- Tail risk: Kozlowski, Veldkamp and Venkateswaran (2017)
  - *We add*: riskless rate, liquidity
- Belief shocks: Gourio (2012), Angeletos and La’O (2013), Bloom (2009)...
  - *We add*: endogenous belief revisions, persistence
- Learning models: Johannes et. al. (2012), Cogley and Sargent (2005)...
  - *We add*: production, non-parametric learning
- Endogenous uncertainty: Fajgelbaum et.al. (2014), Straub and Ulbricht (2013)...
  - *We add*: empirical discipline on beliefs, larger effects
The firm’s problem

\[ V(K_t, B_t, S_t) = \max_{X_t, N_t, B_{t+1}, \hat{K}_{t+1}} H(X_t) - X_t + F(K_t, N_t) - W_t N_t + K_t (1 - \delta) \]
\[ + B_t - P_t B_{t+1} - \hat{K}_{t+1} + \beta E_t M_{t+1} V(K_{t+1}, B_{t+1}, S_{t+1}) \]

s.t. \quad X_t \leq B_t + \eta K_t, \quad K_{t+1} = \phi_{t+1} \hat{K}_{t+1}

Optimality conditions:

\[ 1 = \beta E_t \{ M_{t+1} \phi_{t+1} [F_1(K_{t+1}, N_{t+1}) + 1 - \delta + \eta \mu_{t+1}] \} \]
\[ P_t = \beta E_t \{ M_{t+1} (1 + \mu_{t+1}) \} \]
\[ \mu_t = H'(X_t) - 1 \]
Source: CBOE. Constructed from out-of-the-money put options on S&P 500. A level > 100 indicates negative skewness.
### Long-run analysis

#### Model with liquidity ($\eta > 0$)

<table>
<thead>
<tr>
<th></th>
<th>$\hat{g}_{2007}$</th>
<th>$\hat{g}_{2009}$</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln F(K, N)$</td>
<td>2.39</td>
<td>2.36</td>
<td>-0.03</td>
</tr>
<tr>
<td>$\ln X$</td>
<td>2.68</td>
<td>2.65</td>
<td>-0.03</td>
</tr>
<tr>
<td>$\ln K$</td>
<td>4.10</td>
<td>4.06</td>
<td>-0.04</td>
</tr>
<tr>
<td>Riskless rate ($R^f$), in %</td>
<td>2.31</td>
<td>0.86</td>
<td>-1.45</td>
</tr>
<tr>
<td>Return on capital ($R^\nu$) in %</td>
<td>5.30</td>
<td>5.29</td>
<td>-0.01</td>
</tr>
<tr>
<td>Premium ($R^\nu - R^f$) in %</td>
<td>2.99</td>
<td>4.43</td>
<td>1.44</td>
</tr>
</tbody>
</table>

#### Model without liquidity ($\eta = 0$)

<table>
<thead>
<tr>
<th></th>
<th>$\hat{g}_{2007}$</th>
<th>$\hat{g}_{2009}$</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln F(K, N)$</td>
<td>2.27</td>
<td>2.19</td>
<td>-0.09</td>
</tr>
<tr>
<td>$\ln X$</td>
<td>1.29</td>
<td>1.29</td>
<td>0.00</td>
</tr>
<tr>
<td>$\ln K$</td>
<td>3.93</td>
<td>3.80</td>
<td>-0.13</td>
</tr>
<tr>
<td>Riskless rate ($R^f$) in %</td>
<td>2.31</td>
<td>2.29</td>
<td>-0.02</td>
</tr>
<tr>
<td>Risky return ($R^\nu$) in %</td>
<td>5.28</td>
<td>5.27</td>
<td>-0.01</td>
</tr>
<tr>
<td>Risk premium ($R^\nu - R^f$) in %</td>
<td>2.97</td>
<td>2.98</td>
<td>0.01</td>
</tr>
</tbody>
</table>
Interest rates in the long-run, without liquidity effects:

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\hat{g}_{2007}$</th>
<th>$\hat{g}_{2009}$</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.31</td>
<td>2.29</td>
<td>-0.02</td>
</tr>
<tr>
<td>2</td>
<td>2.31</td>
<td>2.23</td>
<td>-0.08</td>
</tr>
<tr>
<td>10</td>
<td>2.31</td>
<td>1.67</td>
<td>-0.64</td>
</tr>
<tr>
<td>Parameter</td>
<td>Value</td>
<td>Description</td>
<td>Target</td>
</tr>
<tr>
<td>-----------</td>
<td>-------</td>
<td>------------------------------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>Preferences:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.95</td>
<td>Discount factor</td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.50</td>
<td>$1$/Frisch elasticity</td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td>1</td>
<td>Labor disutility</td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.5</td>
<td>Risk aversion</td>
<td></td>
</tr>
<tr>
<td>Technology:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.40</td>
<td>Capital share</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.06</td>
<td>Depreciation rate</td>
<td></td>
</tr>
<tr>
<td>Liquidity:</td>
<td></td>
<td>$H(X) = 2\zeta\sqrt{X - \xi}$</td>
<td></td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.16</td>
<td>Pledgability of capital</td>
<td>Short term obligations</td>
</tr>
<tr>
<td>$\bar{B}$</td>
<td>4.93</td>
<td>Supply of liquid assets</td>
<td>Liquid assets</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>3.93</td>
<td>Investment technology</td>
<td>Riskless rate</td>
</tr>
<tr>
<td>$\xi$</td>
<td>9.00</td>
<td>Investment fixed cost</td>
<td>Capital-output ratio</td>
</tr>
</tbody>
</table>
Data

- **5-year rate, 5 years forward:**
  Nominal 5y rate, 5 years forward from Treasury yield curve
  Expected 5y inflation, 5y forward, from Cleveland Fed 5y and 10y exp inflation

- **Expected returns** $\mathbb{E}(R^e)$: Follow Cochrane (2011) and Hall (2015)
  Regress 1y S&P return to log of the ratio of the S&P to its dividends and log of the ratio of consumption to disposable income forecast model

- **Third moment:**
  \[ SKEW_t = 100 - 10 \frac{\mathbb{E}(R^e - \bar{R}^e)^3}{(VIX_t/100)^3} \]

- **Tail probabilities:** Approximate distribution for $\omega = \frac{x-\mu}{\sigma}$:
  \[ f(\omega) = \varphi(\omega) \left[ 1 - \gamma \frac{(3\omega - \omega^3)}{6} \right] \]
  where \[ \gamma = \mathbb{E} \left[ \frac{x-\mu}{\sigma} \right]^3 \]
Why shocks to capital ‘quality’?

- Most direct way to generate large, negative capital returns + transparent measurement
- Price changes tied to productive value → without persistence, countercyclical investment
  - E.g. discount factor induced price changes ruled out

Source: Prologis Research
• Concern: Methodological changes in the FoF for valuing non-financial assets
• Issue: Consistently measured data series available only for shorter samples
• Strategy: Use NCREIF Property Index for comparison
What if the learning sample includes pre-1950 data?

- **Concern**: Effect of new observations with a longer sample? Great Depression?
- **Issues**: Data availability? Discounting of old data?
- **Strategy**: Use the 1950-2009 sample as a proxy for 1890-1949
  - Great Depression: \( \{\phi_{1929}, \phi_{1930}\} = \{\phi_{2008}, \phi_{2009}\}^\varepsilon, \quad \varepsilon \in \{1, 2\} \)
  - Weights: Observation in \( t - s \) is given a weight \( \lambda^s, \quad \lambda \leq 1 \)
- **Exercise I**: Simulate by drawing from \( \hat{g}_{2009} \)

<table>
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<tr>
<th>Parameters</th>
<th>Long-run Average</th>
<th>Chg</th>
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<tbody>
<tr>
<td>(\varepsilon)</td>
<td>(\lambda)</td>
<td>(\hat{g}_{2007})</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1.68</td>
</tr>
<tr>
<td>1</td>
<td>0.99</td>
<td>2.35</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.22</td>
</tr>
<tr>
<td>2</td>
<td>0.99</td>
<td>2.08</td>
</tr>
</tbody>
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More data (± modest discounting) yields similar results.
What if the learning sample includes pre-1950 data? (contd..)

- Exercise II: Simulate by drawing from $\hat{g}_{2007}$

Similar patterns even with discounting and no more tail events